MATRICULATION MECHANICS

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PREFACE

THE aim of this book is to afford beginners a thorough grounding in those parts of Dynamics and Statics which can be treated without assuming a previous knowledge of Trigonometry.

In order to emphasise the fundamental principles of the subject more fully, the introduction of mathematical formulae has been avoided as far as possible, except where they form an essential feature of the subject. For the same reason, the first ten chapters deal exclusively with the relations between velocity, accleration, mass, and force as applied to motion in a straight line. The Parallelogram Law and its important corollaries are then extended from velecities to forces, and Statics begins where Dynamics ends.

Special attention has been given to the Principle of Moments and its elementary applications to the equilibrium of beams and rods under parallel forces; also to the determination of centres of gravity of "compound" rectilinear and other figures.

For the present edition the type has been reset. The larger sized type has been used for bookwork, the smaller size for notes, explanations, and examples. The more important pieces of bookwork are distinguished by having their section numbers as well as their headings printed in dark type (thus **136**). In the letterpress. letters denoting points in figures are printed thus—**P**, **Q**; those representing algebraic magnitudes, such as the measure of a force, in ordinery italics, thus—*P*, *Q*. Sections and examples marked with an asterisk may be omitted on a first reading of the book.

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CONTENTS.

MECHANICS.

INTRODUCTION.

UNITS.

1. Mechanics Defined: Branches of Mechanics.—The name **Mechanics** was originally used to designate the science of making machines. It is now, however, very generally applied to the whole theory which deals with motion and with bodies acted on by forces.

The subject Mechanics* is generally divided into two parts—

(1) **Dynamics**, which treats of moving bodies;

(2) **Statics**, which treats of bodies kept at rest under the action of forces.

2. By **force** is meant "any cause which changes or tends to change a body's state of rest or motion." In other words, whatever is capable of setting things in motion, or stopping them when they are in motion, or altering the way in which they are moving, is called "force."

Noting that force is defined by means of motion, it is necessary, before considering the properties of force, to consider the properties of motion itself. This branch of the subject is called **Kinematics**.

We then investigate the properties of force as deduced from the properties of motion; this branch is called **Kinetics**.

* There is a little diversity of opinion as to the use of the names Mechanics and Dynamics. Some writers include Statics in Dynamics, thus using the name Dynamics for what we have called Mechanics.

Lastly, in **Statics**, we treat of certain properties of force which do not involve any consideration of motion.

It is thus evident that very little can be said about force until Kinematics has been dealt with; and for this reason we shall not treat of the measurement of force till Chapter VI.

3. **Origin and Uses of Mechanics.**—Mechanics is one of the oldest sciences, for its study originated with the first attempts to make contrivances for raising weights. But it is only within the last three centuries that a simple and consistent theory of the relations of force to matter and motion has been developed. The laws of motion were first discovered by **Galileo** (about the year 1600) from a series of experiments on falling bodies dropped from the top of the leaning tower at Pisa. They were afterwards re-stated by **Newton** in his *Principia* (1687) in the form known as **Newton's Three Laws of Motion**, and as such they are now universally accepted as the basis of Mechanics.

In this book we shall chiefly consider how the laws of motion may be applied to determine the behaviour of given bodies under given forces, not experimentally, but by calculation alone.

4. **Three Fundamental Quantities to be Measured.**—In Mechanics we have to deal with three fundamental notions, namely, **space**, **time**, and **matter**. It is difficult to give an exact definition of these notions, but they are so familiar to us that this is hardly necessary.

The word **space** in Dynamics refers only to one dimension of space, *i.e.*, length, or distance from point to point. It can be measured by a foot-rule or a tape.

A good watch or clock affords the means of measuring **time**, but in Mechanics we are more concerned with duration of time than with the time of day.

A distinction should be clearly kept between an *instant* and an *interval*, the former corresponding to a *point* in Geometry (*punctum temporis*), the latter to a *line* of definite length.

It is more difficult to specify how quantities of **matter** are to be measured, and before we can do so we must clear the way by the following definitions :—

5. Mass: DEFINITIONS.—Quantity of matter is called **mass**. Any limited quantity of matter is called a **body**.

Thus a stone, a piece of earth, wood, or metal, a drop of water, the whole of the Earth's globe, the Sun, and the other "celestial bodies," are all *bodies*.

A **particle** is a body whose size is so small that it may be regarded as a quantity of matter or mass collected at a single point.

6. The usual way of estimating the quantity of matter in a body is by weighing it, *i.e.* placing it in a pair of scales, and balancing it with suitable pieces of metal called "weights" placed in the opposite scale-pan. In the course of the present book it will be shown that *what is commonly called the "weight" of a body gives a correct measure of its mass.*

7. **Units.**—To measure any quantity, whether of length, time, or mass, or anything else, we must first fix on some definite quantity of the same kind, and call this our **unit of measurement.** Having selected this unit, any other quantity will be measured by the *number* of units it contains, such number being called the *measure* of the quantity.

The measurement of quantities in terms of some unit is familiar in every-day life, but the use of the word "unit" in this connection is not so familiar. A few illustrations will make the matter clear. If we speak of a sum of money as (say) *five pounds*, we imply that, taking a *pound* as the unit of money, the number of such units in the sum is 5. Similarly, in speaking of *six yards* of calico, the unit of length is a *yard*, and the number of such units is 6. And by *ten pounds* of sugar we mean that, if the unit of *mass* is a *pound*, the number of such units in the specified quantity of sugar is 10. Notice that the measure 10 pounds specifies the *mass* of the sugar.

The unit of measurement must always be something of the same kind as the quantity to be measured. For measuring a length, we must take some *length* for the unit; for measuring a quantity of matter, we must take some *mass* as our unit. The choice of a unit is, to a certain extent, arbitrary. Certain definite units are very generally adopted, and to these different *names* have been given.

The *measure*, or number which measures any quantity, depends on the unit taken. Thus, 24 pence and 2 shillings represent the same sum of money; when a penny is taken as the unit, the number measuring it is 24, and when a shilling is taken as the unit, the same sum is measured by 2. On the other hand, 2 shillings is not the same as 2 pence. Hence, in specifying a definite quantity of anything (*e.g.*, 2 *shillings*), we must give *two* data :—

(1) *The* name *of the unit chosen* (*in this case shillings*).

(2) *The* number *of units in the quantity measured* (*in this case* 2).

If we left out the word "shillings" and said simply "2," we should leave it quite vague whether we meant 2 shillings, 2 pence, or 2 pounds.

By change of units is meant the same thing as "reduction" in Arithmetic. When we reduce from yards to feet, we are given that a length contains (say) 2 yards, and we have to find its measure in feet (viz. 6). This process we shall call *changing the unit of length from a yard to a foot.*

9. The Foot-pound-second System; or English System.—The most convenient **unit of length** in common use in England is the **foot** (ft.). A foot is one-third of a yard, the yard being defined as the distance between two marks on a certain bar of bronze kept at the Standards office of the Board of Trade in London, at a temperature of 62° Fahr.

Other lengths, such as an inch, a mile, etc., are sometimes taken as units; but in Mechanics it is generally best to measure all lengths in feet.

The **unit of time** is the **mean solar second**, the duration of which is derived from the average length of the solar day (1 day $= 24 \times 60 \times 60$ seconds).

Here, again, we may occasionally use minutes or hours with advantage.

The English **unit of mass** is the **pound avoirdupois** (lb.), and is the mass of a piece of platinum which is preserved in the Standards office. The mass of any other body is one pound if that body will balance the standard mass when placed in a pair of scales.

In some special cases, the ton or ounce avoirdupois, or the Troy grain may be used. For instance, a cubic foot of water is said to contain 1000 ounces, and a metric gramme contains 15·432 grains, of which 7000 make 1 pound avoirdupois.

The system of units based on taking the foot, pound, and second as units of length, mass, and time, respectively, will be spoken of as the **foot-pound-second** or **F.P.S.** system.

9. The Metric and C.G.S. Systems.—The system of weights and measures in common use in France and certain other countries is called the **metric system.**

The **metric unit of length** is the **metre.** It was originally defined as the ten-millionth part of the length of a quadrant of the Earth's circumference measured from the North Pole to the Equator. Thus the whole circumference of the Earth is 40,000,000 or 4×10^7 metres.

The submultiple and multiple units of length are formed by repeatedly dividing or multiplying the metre by 10, as follows, the most important being printed in dark type :—

A metre	=	1000 **millimetres** (mm.).
„	=	100 **centimetres** (cm.).
„	=	10 *decimetres.*
10 metres	=	1 *decametre.*
100 „	=	1 *hectometre.*
1000 „	=	1 **kilometre** (km.).
10,000 „	=	1 *myriametre.*

For scientific purposes the unit of length generally adopted is the **centimetre**, or hundredth of a metre.

The **unit of mass** is the **gramme, or gram** (gm.), and was originally defined as the mass of a cubic centimetre of distilled water at the temperature 4° Centigrade.

Thus, if a small cubical box be made, having its length, breadth, and depth (inside measurement) each one centimetre, and if the box be filled with pure water at the right temperature, the mass of this quantity of water is a gramme.

The submultiple and multiple units derived from the gramme by dividing or multiplying by ten are indicated by the same prefixes as in the case of the metre ; thus :—

A gramme	=	1000 **milligrammes** (mgm.).
„	=	100 *centigrammes.*
„	=	10 *decigrammes.*
10 grammes	=	1 *decagramme.*
100 „	=	1 *hectogramme.*
1000 „	=	1 **kilogramme** (kilog. or kgm.).
10,000 „	=	1 *myriagramme.*

The unit of time is the same in the metric system as in the English. The system of units based on the centimetre as unit of length, the gramme as unit of mass, and the second as unit of time, is called the **centimetre-gramme-second** system, or the **C.G.S.** system, and is used extensively in all countries for mechanical, physical, and electrical measurements.

10. **Advantages of the Metric System.**—From the above description it will be seen that the metric system possesses the following advantages :—

(i) Each unit is exactly ten times the next smaller unit of the same kind, and therefore in changing the unit there is not the tedious multiplication or division required to reduce from one unit to another in the English system—*e.g.* from feet to inches or from ounces to pounds.

(ii) The units of length, volume, and mass are conveniently related. Thus we can write down at once the volume of a quantity of water in cubic centimetres if we know its mass in grammes, and *vice versâ.*

11. **Diagram of the Metric System.**—The large diagram represents a cube whose side is one decimetre, the lengths on its front face being drawn to scale. The large cube would hold a kilogramme of water, while the small cube at the left-hand top corner would hold a gramme of water.

TABLES.

(Not to be committed to memory.)

1. Metric Units of Length.

1 centimetre	=	0·3937079 inches.
1 metre	=	39·37079 "
	=	3·2808991 feet.
1 kilometre	=	3280·8991 "
	=	1093·6330 yards
	=	0·6213 miles.

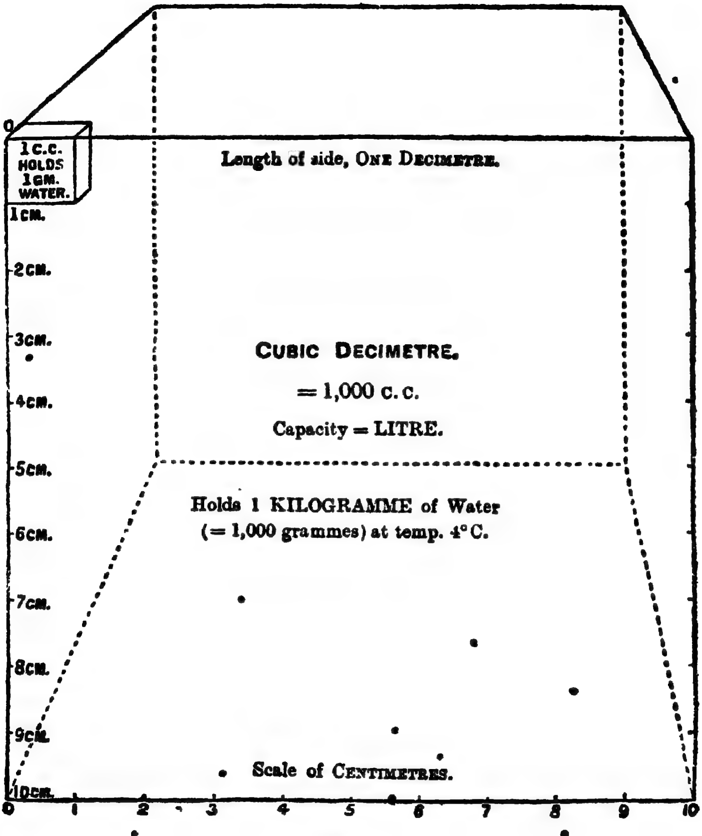
0
1 c.c.
HOLDS
1 GM.
WATER.
Length of side, One Decimetre.
1 CM.
2 CM.
3 CM.
4 CM.
5 CM.
6 CM.
7 CM.
8 CM.
9 CM.
10 CM.
Cubic Decimetre.
= 1,000 c. c.
Capacity = LITRE.
Holds 1 KILOGRAMME of Water
(= 1,000 grammes) at temp. 4° C.
Scale of Centimetres.
0 1 2 3 4 5 6 7 8 9 10

Fig. 1.

2. Metric Units of Mass.

1 milligramme	=	·0154323488 grains.
1 gramme	=	15·4323488 „
	=	·0353739 oz.
1 kilogramme	=	2·20462 lb

3. Velocities.

Velocity of sound in air	=	1,120 feet per second.
„ light	=	186,330 miles per second.
	=	299,860 kilometres per sec.
„ Martini-Henri rifle bullet	=	1,330 feet per second.

4. Intensity of Gravity.

(The numbers represent, in feet and centimetres, *twice* the distance dropped by a falling body during the first second of its motion, at different places at the sea-level.)

Place.	*Ft. per sec. per sec.*	*Cm. per sec. per sec.*
The Equator	32·091	978·10
London	32·191	981·17
Edinburgh	32·203	981·54
The North Pole	32·255	983·11

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5. Densities (approximate).

	Mass of cubic foot in oz.	*Mass of cubic cm in gm.*
Water	1000	1·0
Atmospheric air	1	·001
Mercury	13568	13·568

6. Power.

One horse power = 550 foot-pounds per second
= 7,460,000,000 ergs per second (roughly).

DYNAMICS.

PART I.

VELOCITIES AND ACCELERATIONS.

CHAPTER I.

VELOCITY.

12. By **Kinematics** is meant the study of motion as motion only. Considerations of what is moving or what produces the motion do not enter this branch of the subject.

When a body continues to occupy the same position for any length of time, it is said to be at rest. When its position varies, it is said to be in motion. Thus motion is *change of position.*

DEFINITION.—**Velocity** is *rate of change of position.*

13. Uniform and Variable Velocity.—The velocity of a moving point or body is said to be *uniform* when the distances which it traverses in equal intervals of time are equal, however short these equal intervals may be.

In other cases the velocity is said to be *variable.*

When velocity is uniform, it is measured by the distance traversed in a unit of time.

The word "per" is used in speaking of a rate. Thus we may restate the above definition of the measure of velocity in the following form :—

Velocity is measured by the distance traversed per *unit time.*

The unit of velocity is the velocity of a body which moves over a unit of length per unit of time.

The F.P.S. unit of velocity is a velocity of one foot per second, and this is now often written in the abbreviated form, 1 ft./sec., or $1\,\frac{\text{ft.}}{\text{sec.}}$, or 1 f.s.

The C.G.S. unit of velocity is a velocity of one centimetre per second, in the new notation 1 cm./sec.

14. To find the distance traversed in any interval of time by a body moving uniformly.

Let v be the velocity of the body; then, by definition, v is the distance traversed in each successive unit of time.

So, in 2 units of time the total distance traversed is $2v$,
in 3 units of time it is $3v$, and so on;
and in t units of time it is tv.

Hence, if s denote the distance traversed in the interval of time whose measure is t, we have

$$s = vt \qquad \text{..........(1)},$$

that is, distance traversed = velocity × time.

Examples.—(1) If the velocity is 88 feet per second, the distance traversed in 25 seconds = 88 × 25 = 2200 feet.

(2) If the velocity is 500 centimetres per second, the distance traversed in a minute (60 seconds) is 60 × 500 or 30,000 centimetres.

(3) To find the number of miles travelled in five minutes with a velocity of 88 feet per second. We cannot put $v = 88$ and $t = 5$ and say "$s = vt = 88 \times 5$," for the velocity 88 is measured in feet per *second* and the time 5 is measured in *minutes.* The formula $s = vt$ is not true unless everything is reduced to one system of units. If we use the foot-second system we must take the time t not as 5 *minutes* but as 300 *seconds.* We then have

$$\text{distance traversed} = 88 \times 300 = 26{,}400 \textit{ feet},$$

because we have taken a *foot* as our unit of length. Reducing this to miles, we find distance traversed = 5 miles.

15. From (1) we have by division

$$v = \frac{s}{t};$$

hence the *velocity of a body may be found by dividing the distance traversed by the time taken in traversing it.*

Example.—A cyclist rides from one milestone to the next in $4\frac{1}{2}$ minutes. To find his velocity in feet per second.

The distance traversed is one mile or 5280 feet, and the time taken is $4\frac{1}{2} \times 60$ or 270 seconds; therefore in one second the distance traversed in feet $= 5280 \div 270 = 19\dot{5}$;

$\therefore$ required velocity $= 19\dot{5}$ feet per second.

16. Change of Units.—When a given velocity is expressed in terms of any given units of length and time, the same velocity may be referred to any other system of units by using the method illustrated in the following examples:—

Examples.—(1) To express a velocity of (*a*) one mile per hour, (*b*) 60 miles per hour, in feet per second.

(*a*) A mile contains 5280 feet and an hour contains 3600 seconds. Hence with velocity of one mile per hour

in 3600 seconds the distance traversed is 5280 feet;
$\therefore$ in 1 second „ „ „ $\frac{5280}{3600}$ feet.

Therefore the velocity is represented in feet per second by $\frac{5280}{3600}$, *i.e.* $\frac{22}{15}$.

(*b*) A velocity of 60 miles per hour is 60 times as great, and it is therefore represented in feet per second by $\frac{22}{15} \times 60$ or 88.

(2) When a foot and a second are the units of length and time, the measure of a certain velocity is 27. What is its measure when a yard and a minute are the units?

With a velocity of 27 ft. per sec., 27 ft. are passed over in a second, and, therefore, 27×60 ft. are passed over in a minute;
i.e., 27×20 yards are passed over in a minute;
$\therefore$ a velocity of 27 ft. per sec. = a velocity of 540 yards per minute;
$\therefore$ the measure of the velocity is 540 when estimated in terms of the new units.

Observation.—The student will find it useful to remember the relation

60 miles per hour = 88 feet per second ....(2).

17. Positive and Negative Velocities.—Where we are dealing with a number of motions in a straight line, some of which motions are in the reverse direction to others, it is convenient to regard velocities in one direction as positive and velocities in the opposite direction as negative, the measures of the latter velocities being negative quantities. A similar convention is also made with reference to the distance traversed, which is considered positive if a body has moved in one direction, and negative if it has moved in the reverse direction, the positive direction being the direction in which it would move with the positive velocity. *With these conventions the equation $s = vt$ always holds true.*

The **velocity** of a body is always to be taken as defining both the rate at which it is travelling and the direction in which it is going.

The term **speed** is, however, often used to denote rate of motion considered without reference to direction.

Thus if we take the positive direction to be from left to right, the velocity of a body moving from left to right at the speed of 3 feet per second will be represented by 3, but the velocity of a body moving from right to left at the same speed will be represented by -3.

Again, if the body has moved 5 feet from left to right the distance traversed will be represented by 5, if it has moved 2 feet to the left the distance traversed will be represented by -2.

18. Representation of Direction by the Order of Letters.—In future, when we speak of "the straight line **AB**," we shall imply that the line is drawn *from* **A** *to* **B**, not from **B** to **A**. If we use the signs $+$ and $-$ to denote directions, as in § 17, the distance **AB** is considered positive if we have to go in the positive direction to get from **A** to **B**, negative if we have to go in the reverse direction.

19. Relative Velocity.—Definition.—By the velocity of one body **relative** to another is meant the rate at which the first body is changing its position with respect to the second.

The meaning and importance of relative velocity will be best understood from the following simple illustrations :—

(1) Suppose that a man on board a large steamer is walking along the deck. We naturally say that he is in motion, because he is walking.

But this motion along the deck is only a *relative motion*, and is not the true motion which he possesses, for he is also at the same time being carried forward by the motion of the steamer. And if the man remains standing in the same part of the deck we know that he is not really at rest, but that he is moving with the steamer.

(2) If a fast train overtakes and passes a slow train, a passenger in the former will obtain the impression that the latter is going backwards, because he is going faster and leaves it behind. This apparent backward velocity is the *relative velocity* of the slow train with respect to the fast. The slow train is really moving forwards all the time, but as it is not going fast enough to keep up with the fast one, it appears to go in the opposite direction to the latter.

20. **Motion Relative to the Earth.**—We are accustomed to consider the Earth as fixed, because we see on it trees, houses, hills, and other objects which always appear to retain the same relative positions. And so in measuring velocities we naturally refer them to the Earth. We observe that the Sun, Moon, and stars rise in the east, and set in the west. At first we should naturally say the stars are moving, and that we are at rest. But it is much easier to believe that the Earth (even though it be 8,000 miles in diameter) is moving as a whole, than that the stars, which are separate and distinct bodies, enormously larger than the Earth, and at distances of many billions of miles apart, are all revolving together about the Earth once in a day, so as always to remain in the same configurations. This and other reasons force upon us the fact that it is the Earth which rotates once a day, and not the stars that move. Further, we are taught that the Earth travels round the Sun, describing, roughly, a circle of radius 92 million miles in the course of the year.

In most cases we do not have to take account of the motions of the Earth. The relative motions can be worked out just as if the Earth were fixed. In fact, most of our ideas of motion are based on experiments made with moving bodies on the Earth, and they refer quite as much to relative motion as to actual motion.

21. **Properties of Relative Velocity.**—From the arguments and examples of the last paragraphs, the following properties will be evident.

When two bodies, **A** and **B**, are moving in any manner, the velocity of **B** *relative to* **A** is the velocity with which **B** would appear to move if the observer were moving with the body **A**.

If the bodies are moving in the same straight line, the rate at which the faster body overtakes and passes the slower one is the relative velocity of the former with respect to the latter, and is the *algebraic difference* of their velocities (taken with proper signs, as in § 17).

Let **A** and **B** be trains moving with velocities 50 and 30 miles per hour respectively, in the same direction. **A**'s velocity relative to **B** is 20 (miles per hour), while **B**'s velocity relative to **A** is -20. But if they are moving in opposite directions, their velocities relative to one another are ± 80.

Since $80 = 50 - (-30)$, this relative velocity is the algebraic difference of 50 and -30, the velocities of the trains.

22. Composition of Velocities in One Straight Line.—From the consideration of relative velocities we naturally pass on to cases where the velocity of a body is due to a number of independent relative motions. We may take the following in illustration of such motions :—

Example.—A river is flowing at the rate of 1 mile an hour, and a man can row a boat through still water at 4 miles an hour. To find his rate of progress (i) down stream, (ii) up stream.

Let **B′AB** be the direction of the river, **O** the man's starting-point. Then in one hour the water that was at **O** will have flowed to a point **A** one mile from **O**, and if the man had allowed his boat to drift it would have reached **A**.

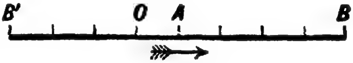

Fig. 2.

But the man has pulled his boat 4 miles relative to the water. Hence, if he is pulling down stream, his action in rowing during the hour will have taken the boat to a point **B** four miles below **A**.

The whole space **OB** is $= 4 + 1 = 5$ miles; hence the man's rate of progress down stream $= 5$ miles an hour.

But if the man pulls up stream, the action of his oars during the hour will take him 4 miles through the water to a point **B′** four miles above **A**.

In this case the whole space **OB′** (measured up stream) $= 4 - 1 = 3$ miles; hence the man's rate of progress up stream $= 3$ miles an hour.

23. Component and Resultant Velocities.—If the motion of a body is due to several simultaneous independent causes, its actual velocity (or change of position per second) is called its *resultant* velocity, the velocities due to the separate constituent causes being called *component* velocities.

Thus, in the example above, the rate of flow of the stream and the rate at which the man rows, relative to the water, are the components

of the velocity of the boat, while his actual rate of progress (relative to a fixed point on the bank) is the resultant velocity.

The process of finding the resultant velocity from the components is called *compounding* velocities. To compound several velocities in the same straight line, we add them together, giving regard to their algebraic signs, as explained in § 17.

24. Variable Velocity.—This may be measured in two ways—

(i) By the *average velocity* in any given interval;

(ii) By the velocity *at* any given instant.

25. Average Velocity: DEFINITION.—The *average velocity* of a moving body in any given interval of time is the velocity with which a body would have to move *uniformly* in order to traverse the same distance in the same time. In other words, the average velocity of a moving body is the whole distance described by the body divided by the whole time of its description. Thus, if v denotes the *average* velocity of a body moving *anyhow* over a space s in a time t, the formula $s = vt$, or $v = s/t$, holds good just as for a uniform velocity.

26. Velocity at any Instant: DEFINITION.—The velocity of a body *at any instant* is measured by the rate per second at which distance is being traversed during a small interval of time containing that instant.

When a *very small* interval is taken, there is no time for any appreciable change of velocity during the interval, so that the velocity may be considered constant throughout the interval, and the formula $v = \frac{s}{t}$ again applies.

Thus, the speed of a railway train may vary considerably in the course of 5 minutes, but in so short an interval, say, as one-tenth of a second there can be no appreciable alteration of its rate.

Hence the term "velocity at any instant" must be regarded as a convenient abbreviation for "average velocity during a very small interval of time including the given instant."

EXAMPLES I.

1. Find the measures of the following velocities, a foot and a second being the units of length and time :—

(i) Ninety miles per hour;
(ii) Twenty yards per minute;
(iii) Two furlongs per half-hour.

2. Find the measures of the following velocities, a yard and a minute being the units of length and time :—

(i) Thirty miles per hour;
(ii) Twenty feet per second;
(iii) Three hundred and sixty feet per hour.

3. Find the measures of the following velocities, when a mile and an hour are the units of length and time :—

(i) Five and a half feet per second;
(ii) Eleven yards per minute;
(iii) One-tenth of a mile per second.

4. A body has a uniform velocity of 10 feet per second; how far will it go in 10 seconds?

5. How far will a train, travelling at the uniform rate of 25 miles an hour, go in half a minute?

6. A body moves with uniform velocity, whose measure is 30 if a yard and a minute be the units of length and time, respectively. How far will it go in 24 seconds?

7. A train moves with uniform velocity, whose measure is 45 if a mile and an hour be the units of length and time respectively. How far will it travel in 10 seconds?

8. A body moves uniformly with a velocity of 30 yards per minute. Find how long it will take to travel 24 feet.

9. A train moves uniformly through a distance of 11 feet in 4 seconds. Find its velocity in miles per hour.

10. Find, in feet per second, the average velocity of a bicyclist who rides a mile in 3 minutes 18 seconds.

11. A three-mile race was run in 15 minutes 24 seconds. Find the average velocity of the winner.

12. Find the relative velocity of two railway trains, travelling at the rates of 30 miles an hour and 50 feet per second, respectively, when they are moving (i) in the same direction, (ii) in opposite directions.

13. A man rows a boat at the uniform rate of 5 miles an hour on a stream which is flowing at the rate of 1 mile an hour. How long will it take him to row 12 miles up-stream, and back?

14. Two trains, 210 yards and 230 yards long, respectively, are travelling in opposite directions, the former at the rate of 25 miles per hour, and the latter at the rate of 35 miles per hour. Find how long it takes them to pass each other.

15. If u be the measure of a velocity in yard-minute units, what is its measure in foot-second units?

16. Find the measures of the following velocities, a metre and a minute being the units of length and time:—

(i) Five centimetres per second;

(ii) Twenty-five kilometres per hour.

17. Find the measures of the following velocities, if a kilometre and an hour are the units of length and time:—

(i) Thirty centimetres per second;

(ii) Five hundred metres per minute.

18. A man walks at the uniform rate of 6 kilometres per hour. How far will he walk in 15 seconds?

19. A body moves with uniform velocity, whose measure is 120 if a metre and a minute are the units of length and time. How far will it go in 5 seconds?

20. Compare the velocities of two bodies, one of which moves at the rate of m feet in n seconds, and the other at the rate of p miles in q hours.

CHAPTER II.

ACCELERATION.

27. Having defined the velocity of a body *at* any instant in § 26, we shall always in future take this as the measure of a variable velocity (unless average velocity is expressly specified). We shall now consider the rate at which this velocity varies from one instant to another.

DEFINITION.—**Acceleration** is *rate of change of velocity*.

Strictly speaking, the word "change" may imply an alteration in direction as well as in magnitude or speed; but in the first two parts of this book we shall be concerned entirely with motion in a straight line, and change of motion in the same straight line. Thus we may say that acceleration is the rate of *increase* of velocity. When there is a *retardation*, or *decrease* of velocity, this is spoken of as a negative acceleration. (See also § 137, forward.)

28. **Uniform and Variable Acceleration.**—Acceleration is said to be *uniform* when the velocity increases by equal amounts in equal intervals of time, however small. In other cases it is said to be *variable*.

With this latter kind we shall have nothing to do.

Uniform acceleration is measured by the amount by which the velocity increases per unit of time, or, practically

acceleration = velocity *added on* per second.

The **unit of acceleration** is the acceleration of a body which moves so that the measure of its velocity increases by unity in a unit of time.

In the F.P.S. system of units, where the unit of velocity is a velocity of one foot per second, the unit of acceleration is the acceleration which in one second increases the velocity by one foot per second, and this is called an acceleration of

"one foot per second per second," written, in the new notation, 1 ft./sec.^{2}, or sometimes 1 f.s.s.

Similarly, the C.G.S. unit of acceleration is an acceleration of "one centimetre per second per second"; in the new notation, 1 cm./sec.^{2}.

OBSERVATION.—The words "per second" must be repeated because the unit of time is involved twice, firstly in measuring the velocity or change of velocity, and secondly in measuring the interval in which this change of velocity takes place.

Examples.—(1) If a body is moving at the rate of 5 feet per second at any instant, and its velocity one second later is 7 feet per second, the increase of velocity in one second is 2 feet per second, and therefore the acceleration is 2 feet per second per second.

(2) If in one second the velocity changes from 10 feet per second to 8 feet per second, the increase of velocity is $= 8 - 10 = -2$ feet per second, and the acceleration is -2 feet per second per second.

(3) Similarly, if the velocities at intervals of one second are 53, 59, 65,.. centimetres per second, the acceleration is 6 centimetres per second per second.

29. To find the velocity acquired in any interval of time by a body moving with uniform acceleration.

Let f be the given acceleration, and let it be required to find the velocity acquired after t units of time have elapsed since the commencement of its action.

(i) Suppose that the moving body starts from rest. Then the velocity acquired in 1 unit of time $= f$,

the velocity acquired in 2 units of time $= 2f$,

and the velocity acquired in t units of time $= tf$.

Hence, if v denote the required velocity,

$$v = ft \quad \text{..................................(1).}$$

(ii) Suppose that the body starts with initial velocity u.

Then, as before, the amount by which the velocity *increases* in t units of time $= ft$;

$\therefore$ the velocity at the end of the interval $= u + ft$;

$\therefore$ in this case $\quad v = u + ft$ ....(2).

COR.—Since $v - u = ft$, we see that

increase of velocity = acceleration $\times$ time.

Examples.—(1) A train acquires a velocity of 60 miles an hour in two minutes. To find its acceleration in F.P.S. units.

In 2 min. (= 120 sec.) the velocity increases by 60 miles per hour
= 88 feet per second;

$\therefore$ in one second the velocity increases $\frac{88}{120}$ foot per second.

Therefore the given acceleration is $\frac{88}{120}$ or $\cdot 7\dot{3}$ foot per sec. per sec.

(2) If the acceleration is 32 feet per second per second, and the body starts with the velocity 100 feet per second, the velocity after ten seconds $= u + ft = 100 + 32 \times 10 = 420$ feet per second.

30. Change of Units.—When an acceleration is expressed in terms of one system of units, we may reduce it to any other system of units, by adopting the method illustrated in the following examples:—

Examples.—(1) To express an acceleration of 32 feet per second per second in yards per minute per minute.

Here we are given that the increase of velocity in one second is 32 feet per second. In order to change to the new units we must

(1) find the increase of velocity acquired in one minute;

(2) express this increase of velocity in yards per minute.

We accordingly proceed as follows:—

In 1 sec. total increase of velocity = 32 feet per second;

$\therefore$ in 1 minute (60 sec.) the total increase of velocity

$$= 32 \times 60 = 1920 \text{ feet per second}$$
$$= 1920 \times 60 \text{ feet per minute}$$
$$= \frac{1920 \times 60}{3} \text{ yards per minute}$$
$$= 38400 \text{ yards per minute.}$$

$\therefore$ given acceleration = 38400 yards per minute per minute.

(2) A body is moving with an acceleration of 54000 miles per hour per hour. Express this in feet per second per second.

A velocity of 54000 miles per hr. = a vel. of $\frac{54000 \times 5280}{60 \times 60}$ ft. per sec.

= a vel. of 1800×44 ft. per sec.

This velocity is gained every hour;

$\therefore$ the gain per second is $\frac{1800 \times 44}{60 \times 60}$ ft. per sec. = 22 ft. per sec.;

therefore an acceleration of 54000 miles per hour per hour
= an acceleration of 22 ft. per sec. per sec.

31. Retardation.—When the speed of a body is decreasing, the motion is said to be retarded. As said before (§ 27), a retardation of a positive velocity is regarded as a negative acceleration.

More generally, if a motion, or change of motion, in one direction be taken as positive, then any motion or change in the opposite direction is considered negative.

(A negative acceleration does not always retard. When a stone is thrown upwards, we take the upward direction as positive, and the acceleration of gravity (see § 52) is negative, and at first retards the speed of the stone, but after it has reached the highest point, the same acceleration of gravity increases the speed.)

Examples.—(1) A body starts with velocity 144 feet per second, and is subject to a retardation of 32 feet per second per second. To find its velocity after 5 seconds.

Here $u = 144$, $f = -32$, $t = 5$; whence, substituting in the formula $v = u + ft$, we have

$$v = 144 + (-32)\times 5 = 144 - 160 = -16.$$

Hence the body is moving with speed 16 feet per second in the *opposite* direction to that in which it started.

(2) If a railway train moving at 60 miles per hour (88 feet per second) is brought to rest in one minute (60 sec.), and we consider the original velocity positive, the acceleration $f = (v - u)/t$

$$= (0 - 88)/60 = -88/60 = -22/15 \text{ feet per sec. per sec.},$$

and is negative. If this acceleration were continued for another minute, the train would acquire a velocity of -88 feet per second, that is, its original velocity would be exactly reversed.

32. Variable Acceleration is measured in a very similar way to variable velocity. It may be measured either

(i) By the *average acceleration* in any given interval;

(ii) By the acceleration *at* any given instant.

EXAMPLES II.

1. How is the measure of an acceleration changed if
   (i) the unit of length be changed from a foot to a yard,
   (ii) the unit of time be changed from a second to a minute?

2. If the measure of the acceleration due to gravity be 32 when a foot and a second are the units of length and time, what will it be when the units are
   (i) a yard and a minute,
   (ii) a mile and an hour?

3. If an acceleration be expressed by 480 when a yard and a minute are the units of length and time, find its measure when a foot and a second are the units.

4. If 110 be the measure of an acceleration when a yard and a second are the units of length and time, find its measure when a mile and a minute are the units.

5. A train which is uniformly accelerated starts from rest, and at the end of 3 seconds has a velocity with which it would travel through a mile in the next 5 minutes. Find the acceleration.

6. How long will it take a body, moving from rest, to acquire a velocity of 400 feet per second if it be uniformly accelerated at the rate of 25 feet per second per second?

7. A body moves from rest with uniform acceleration, and its velocity at the end of 3 seconds is 21 feet per second. Find its velocity at the end of 8 seconds.

8. A body moves at one instant with a velocity of 20 feet per second, and, 6 seconds after, it is moving with a velocity of 50 feet per second. Find the acceleration, supposing it to be uniform.

9. What is the velocity of a body which began to move with a velocity of 100 feet per second, and has continued in motion for 6 seconds, the velocity regularly diminishing at the rate of 10 feet per second per second?

10. A body, which is uniformly accelerated, starts with a velocity of 40 feet per second, and 8 seconds later is moving with a velocity which would carry it through 60 miles in the next hour. Find its acceleration.

11. A point moves from rest with uniform acceleration, and the velocity at the end of $3\frac{1}{2}$ minutes is 70 feet per second. Find the acceleration.

12. A body starts with a velocity of 150 feet per second, and at the end of each second it instantaneously loses $\frac{1}{5}$ of its original velocity. What space will it describe before coming to rest?

13. A body is moving with a velocity of 30 miles an hour, and, 5 minutes after, it is moving with a velocity of 8 yards per second. Find the acceleration, supposed uniform.

14. Express the C.G.S. unit of acceleration, and the acceleration of gravity (980 cm. per sec. per sec.),

(i) in metres per minute per minute,

(ii) in kilometres per hour per hour.

15. If the measure of an acceleration be 60 when a mile and an hour are the units of length and time, find its measure when a centimetre and a second are the units. (Take 1 ft. = 30 cm.)

16. In half a minute the velocity of a body increases uniformly from 10 centimetres per second to 36 kilometres per hour. Find the acceleration.

17. A body is moving at a certain instant with a velocity of 1·8 kilometres per hour, and its velocity is uniformly diminishing at the rate of 5 centimetres/second^2. After how many seconds will the velocity of the body be 15 metres per minute?

EXAMINATION PAPER I.

1. Distinguish between *variable* and *uniform* velocity. Explain how they are measured.

2. Explain what is meant by the *acceleration* of a point moving in a straight line.

3. If 15 be the measure of a velocity when a mile and an hour are the units of length and time, find its measure when the units are a yard and a minute.

4. Compare the velocities of two points which move uniformly, one through 900 feet in one minute and the other through 22½ miles in one hour.

5. A mill-sail is 14 feet long and revolves uniformly 12 times in a minute. Find the velocity of the extremity of the sail.

6. Give a short account of the metric system of weights and measures.

7. The measure of an acceleration is 40,500 when a mile and an hour are the units of length and time. Find its measure when a foot and a second are the units.

8. A body is moving at a certain instant with a velocity of 40 yards per minute, and 10 seconds later it is moving with a velocity of 22 feet per second. Find the acceleration.

9. If the acceleration due to gravity be 32 when a foot and a second are the units of length and time, find its measure when a kilometre and a minute are the units. (Take 1 kilometre = 3280 feet approx.)

10. The velocity of a train diminishes uniformly from 60 miles an hour at 12 o'clock to 10 miles an hour at 25 minutes past 12. What was its velocity at 15 minutes past 12?

CHAPTER III.

UNIFORMLY ACCELERATED MOTION.

33. Preliminary Observations.—In Chapter I. we have shown that, for motion with *uniform velocity* v,

$$s = vt;$$

and in Chapter II. we have shown that, for motion under *uniform acceleration* f, the velocity *at any instant* is given by

$$v = u + ft.$$

We have now to find expressions for the distance traversed in any time-interval t by a body moving with uniform acceleration. For, in accelerated motion, the velocity is variable, and $s = vt$ does not hold good.

Illustration.—If a railway train starts from rest with uniform acceleration, and at the end of one minute it has acquired a velocity of a mile a minute, the train has *not travelled a mile in that minute*. For to do so it would have to go at full speed the whole time, but in reality the train never acquires this speed till the end of the minute; at the beginning of the minute it is not moving at all.

At the middle of the interval, or half-a-minute from starting, the velocity is $\frac{1}{2}$ a mile per minute, and in each second it increases by $\frac{1}{60}$ of a mile per minute. One second before the middle of the interval the velocity is $\frac{1}{60}$ of a mile per minute less than at the middle, and one second after the middle it is $\frac{1}{60}$ of a mile per minute greater. And generally, the velocity t seconds before the middle of the interval is as much below $\frac{1}{2}$ a mile per minute as the velocity t seconds after is above. Hence we are led to assume that the average velocity is $\frac{1}{2}$ a mile per minute, and that the distance traversed in the minute is $\frac{1}{2}$ a mile. The distance traversed in the first half-minute is, of course, less than $\frac{1}{2}$ of half a mile, and the distance traversed in the second half-minute is more than $\frac{1}{2}$ of half-a-mile by an equal amount; but the two together make up exactly one half-mile.

34. To find the space described, under a uniform acceleration f, in a time-interval t seconds.

Let u be the velocity at the beginning of the interval,

v the velocity at the end.

Divide the time t into n equal parts, each of length i, where n is a large even number. We thus have

$$t = ni, \quad \text{or} \quad i = t/n.$$

By making n very large, the intervals i will be very small, so that the velocity in one of these will have no time to alter appreciably, and we may consider the velocity at the beginning or the end of an interval to represent the velocity throughout that interval. Thus we have

velocity at beginning of 1st interval $= u$,

" " " 2nd " $= u + fi$,

" " " 3rd " $= u + 2fi$,

etc., etc.

Also velocity at end of last interval $= v$,

" " last but 1 $= v - fi$,

" " last but 2 $= v - 2fi$,

etc., etc.

The distance traversed in any interval is found by multiplying the corresponding velocity by i. Now take the small intervals in pairs, and combine *the first interval with the last, the second with the last but one*, and so on, the $(m + 1)$th interval being combined with the last but m.

Then sum of distances traversed

in 1st and last intervals $= (u + v)i$,

in 2nd and last but 1 $= (u + fi + v - fi)i = (u + v)i$,

in 3rd and last but 2 $= (u + 2fi + v - 2fi)\, i = (u + v)i$,

in $(m+1)$th and last but $m = (u + mfi + v - mfi)i = (u + v)i$.

Therefore the distance traversed in each pair of intervals i is $(u + v)\, i$, and is the same as if the velocity in the pair were $\frac{1}{2}(u + v)$. And, since this is true of every pair, the whole distance traversed is the same as if the velocity were $\frac{1}{2}(u + v)$ throughout the whole of the time t.

Therefore distance traversed $= \frac{1}{2}(u + v)t$.

Thus the distance traversed under uniform acceleration is the product of the time into half the sum of the initial and final velocities.

COR. Hence average velocity $= s/t = \frac{1}{2}(u + v)$.

35. Alternative Proof.—Let u and v be the first and last velocities, t seconds the time of motion. Divide the time into a number n of very small intervals, each equal to i. If n is very great, the velocity during any one of these intervals may be considered constant. Owing to the *uniformity* of the acceleration, the successive velocities in these intervals form a series in Arithmetical Progression, and so do the small spaces (ui, etc.) described in them. Now, by the formula for the sum of an A.P.,

$$s = \tfrac{1}{2}n(a + l),$$

where a, l are the first and last terms.

Putting $a = ui$, $l = vi$, we have for the sum of the spaces described in all the intervals (*i.e.* the whole space)

$$\begin{aligned} s &= \tfrac{1}{2}n(ui + vi) = \tfrac{1}{2}(u + v)\, i \times n \\ &= \tfrac{1}{2}(u + v)\, t. \end{aligned}$$

Example.—A train, starting from rest, acquires a velocity of 48 miles an hour in $2\frac{1}{2}$ minutes; to find the distance run in that time.

Here the initial velocity is zero, and the final velocity is $\frac{4}{5}$ mile per minute. Therefore the average velocity is $\frac{2}{5}$ mile per minute, and the distance run in the $2\frac{1}{2}$ minutes

$$= \tfrac{2}{5} \times 2\tfrac{1}{2} = 1 \text{ mile}.$$

36. Uniformly Accelerated Motion from Rest.

The formulae for uniformly accelerated motion, which we shall now deduce, are very important.

Let f be the uniform acceleration,

t the time, measured from the instant of rest,

v the velocity acquired at the end of the time t,

s the distance traversed in the time t.

Putting $u = 0$ in the two formulae

$$v = u + ft \quad \text{and} \quad s = \tfrac{1}{2}(u + v)\,t,$$

we have

$$v = ft \quad \text{.................................(1)},$$

$$s = \tfrac{1}{2}vt \quad \text{.................................(2)}.$$

Eliminating v from these by multiplying them together,

$$s = \tfrac{1}{2}ft^2 \quad \text{.................................(3)}.$$

Again, eliminating t from (1) and (2) by division, we get

$$v^2 = 2fs \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Formulae (1), (2), (3), (4) are sufficient to work out any problem relating to uniformly accelerated motion from rest.

37. Uniformly Accelerated Motion with an Initial Velocity.— We have proved, in § 29, that

$$v - u = ft, \quad \text{or} \quad v = u + ft \ldots\ldots\ldots\ldots\ldots\ldots(5);$$

and, in § 34, that

$$s = \tfrac{1}{2}(v + u)\, t \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6);$$

and from these two equations we may find a relation between any four of the quantities u, v, f, t, s. Thus, eliminating v, we have

$$s = \tfrac{1}{2}(u + u + ft)\, t,$$

or

$$s = ut + \tfrac{1}{2}ft^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

giving the distance traversed in terms of the time, the *initial* velocity, and the given acceleration.

Lastly, on eliminating t by multiplying (5) and (6) across, we have

$$(v - u)\times\tfrac{1}{2}(v + u) = fs, \quad \text{or} \quad \tfrac{1}{2}(v^2 - u^2) = fs,$$

or

$$v^2 - u^2 = 2fs, \qquad \text{or} \qquad v^2 = u^2 + 2fs \ldots\ldots(8),$$

a relation between the distance traversed and the initial and final velocities.

The average velocity in any interval $= s/t$

$$= u + \tfrac{1}{2}ft = u + f(\tfrac{1}{2}t)$$
$$= \text{velocity at time } \tfrac{1}{2}t$$
$$= \text{vel. at middle of interval.}$$

38. **The signs of the letters should never be changed in the formulae,** even when we are dealing with a retardation. In such a case, the *value* of f is a *negative* quantity, but the *formula* (7) must still be written

$$s = ut + \tfrac{1}{2}ft^2, \quad \text{and not} \quad s = ut - \tfrac{1}{2}ft^2.$$

The following examples will show how the formulae may be applied to retarded motion :—

Examples.—(1) If a steamer starts from rest with an acceleration of 100 yards per minute per minute, it will at the end of five minutes have attained a velocity equal to 100×5 yards per minute.

$$(v) \quad = \quad (f)\times(t)$$

(2) Now suppose that, when the steamer is going at the rate of 500 yards per minute, the engines are reversed, so as to produce a *backward* acceleration of 100 yards per minute per minute, and let it be required to find how far the steamer will go in 3 minutes.

We must now put $f = -100$; and therefore

our formula $\qquad s = ut + \frac{1}{2}ft^2$

gives us $\qquad s = 500 \times 3 + \frac{1}{2}(-100) \times 3^2$;

i.e. $\qquad s = 1500 - 450$,

or $\qquad$ distance traversed $= 1050$ yards.

(3) If a train, when going at 50 miles an hour, can be pulled up in 48 seconds, find at what point the brakes must be applied.

When the train is being pulled up, the initial velocity is $\frac{5}{6}$ mile per minute, and the final velocity is zero. Also the time taken in pulling up equals $\frac{4}{5}$ of a minute.

Therefore the distance run when the brakes are on

$$= \tfrac{1}{2}(u + v)\,t = \tfrac{5}{12} \times \tfrac{4}{5} = \tfrac{1}{3} \text{ of a mile.}$$

Hence the brakes must be applied when the train is $\frac{1}{3}$ of a mile from the station.

(These examples show that it is not always necessary to reduce to feet and seconds.)

39. To find the distance traversed in the nth second of a body's motion.

With the usual notation, taking the second as the unit of time,

$$\text{velocity at end of } n - 1 \text{ seconds} = u + f(n - 1),$$

$$\text{velocity at end of } n \text{ seconds} = u + fn;$$

$\therefore$ average velocity during nth second

$$= \tfrac{1}{2}\{u + f(n - 1) + u + fn\} = u + \tfrac{1}{2}(2n - 1)f;$$

and, since the measure of a second is unity, the distance traversed in the nth second

$$= \{u + \tfrac{1}{2}(2n - 1)f\} \times 1 = u + \tfrac{1}{2}(2n - 1)f.$$

It is better, however, to remember the method by which this formula is obtained, and not the formula itself.

40. To find the acceleration of a moving body by observation, it is only necessary to observe the distances traversed in two successive seconds of the motion.

Examples.—(1) If the distances traversed in two successive seconds are 10 feet and 42 feet, to find the acceleration, supposing it uniform.

Here vel. at mid. of 1st sec. = av. vel. in 1st sec. = 10 ft. per sec.,
" " 2nd sec. = " " 2nd sec. = 42 ft. per sec.

Therefore, from middle of 1st to middle of 2nd second of time, velocity increases from 10 to 42 feet per second ;

$\therefore$ increase of velocity in 1 sec. = 32 ft. per sec. ;

$\therefore$ acceleration = 32 ft. per sec. per sec.

(2) A body traverses altogether 66 feet in the fifth, sixth, and seventh seconds of its motion from rest under uniform acceleration. To find the value of this acceleration.

The average velocity in the three seconds

$$= \tfrac{66}{3} = 22 \text{ ft. per sec.}$$

This is the velocity in the middle of the interval ; *i.e.* $5\frac{1}{2}$ seconds after starting ;

$$\therefore \text{ the acceleration} = \frac{22}{5\frac{1}{2}} = 4 \text{ ft. per sec. per sec.}$$

41. Graphic Representation of Variable Velocity.—We shall now show how motion with variable velocity can be fully represented by drawing a curve which serves as a sort of map or diagram of the velocity (Fig. 3).

Take a straight line **OX** (which we will suppose horizontal), and, having selected any point **O** on it, measure a length **OM**, such that the number of units of length in **OM** is equal to the number of units of time (say seconds) that have elapsed since the beginning of the motion : thus, if **OM** contains *t* units of length, the point **M** will represent the time *t*. The points *a*, *b*, *c*, distant respectively 1, 2, 3 units of length from **O**, will represent the times 1, 2, 3 seconds after the beginning of the motion, respectively.

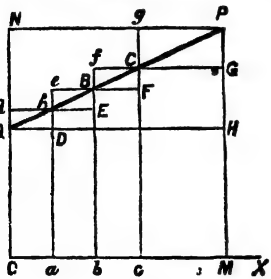

Fig. 3.

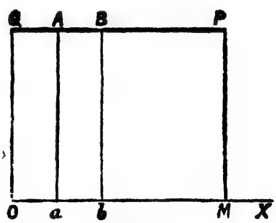

Fig. 3*a*.

Through **M** draw a line **MP** perpendicular to **OM**, and let the number of units of length in **MP** be equal to the number of units of velocity in the velocity of the moving point at the instant represented by **M**. Let

similar perpendiculars be erected at every point on **OX**, so that (for example) a**A**, b**B**, c**C** . . . are to be taken proportional to the velocities at the times 1, 2, 3, . . . seconds, respectively. Then the extremities of these perpendiculars will be found to lie along a certain straight or curved line **ABCP**. This line may be called the **velocity graph** of the motion.

I. When the motion is uniform, the velocity graph is a straight line parallel to **OX**; for, if the velocity is u, all the "ordinates," such as **MP**, are u units long, and therefore the points on the velocity graph are at the same distance from **OX**. In this case, if t be the time **OM** (Fig. 3a), we have

$$\text{distance traversed} = ut = \mathbf{MP} \times \mathbf{OM} = \text{area of rectangle } \mathbf{OP}.$$

42. We shall now extend this result to variable velocities by showing that—

II. The distance traversed in any interval of time is represented by means of the area contained by the velocity graph and the two bounding ordinates.

Let **OM** represent the given interval, **QP** the velocity graph (Fig. 3); then it is required to show that the area **OQPM** represents the distance traversed.

Divide **OM** into any number of intervals at the points a, b, c, and draw the ordinates a**A**, b**B**, c**C** to meet the velocity graph in **A**, **B**, **C**, so that **OQ**, a**A**, b**B**, c**C**, **MP** represent the velocities at the instants of time represented by **O**, a, b, c, **M**.

If the velocity during each of the intervals **O**a, ab, bc, c**M** were uniform and equal to the actual velocity at the *beginning* of that interval, the distances traversed in the intervals would be **OQ** . **O**a, a**A** . ab, b**B** . bc, etc., and would be represented by the measures of the areas of the rectangles **Q**a, **A**b, **B**c, etc., and the whole distance traversed would be represented by the sum of the measures of these rectangles; that is, by the area of the inscribed figure **OQDAEBFCGMO**.

In like manner, if the velocity throughout each interval were equal to the actual velocity at its end, the distance traversed would be represented by the area of the circumscribing figure **O**d**A**e**B**f**C**g**PMO**.

Now however small may be the distances **O**a, ab, bc, . . . into which **OM** is divided the above argument still holds. Moreover, when these distances are very small the inscribed and circumscribed figures are both practically equal to the figure **OQPMO**. This area may therefore be taken to represent the actual space described.

III. To prove graphically the formula for uniformly accelerated motion, $s = ut + \frac{1}{2}ft^2$.

If the velocity graph **QP** (Fig. 3) is a straight line, and if **O**a, ab, bc be equal, it is evident that **DA** = **EB** = **FC** . . . Hence the velocity increases by equal amounts (**DA**, **EB**, **FC** . . .) in equal times (**O**a, ab, bc . . .). That is to say the acceleration is uniform.

Conversely it can be proved that if the acceleration is uniform the velocity curve is a straight line.

Let **OQ** denote the initial velocity u, **OM** the time t, and **MP** the final velocity $u + ft$.

Then $$MH = OQ = u,$$

and therefore $HP = ft$; and we have

$$\begin{aligned} \text{distance travelled} &= \text{area } OMPQ = \text{rect. } OMHQ + \triangle\, QHP \\ &= \text{rect. } OMHQ + \tfrac{1}{2}\,\text{rect. } QHPN \\ &= OQ\,.\,OM + \tfrac{1}{2}HP\,.\,OM \\ &= u\,.\,t + \tfrac{1}{2}ft\,.\,t = ut + \tfrac{1}{2}ft^2. \end{aligned}$$

Example.—A body, starting from rest, travels north for 6 seconds. The velocities at the end of successive seconds are as follows:—After 1 second, 30 ft./sec.; after 2 sec., 55 ft./sec.; after 3 sec., 65 ft./sec.; after 4 sec. 55 ft./sec.; after 5 sec., 30 ft./sec.; after 6 sec., zero. If the change of velocity is at no time abrupt, determine roughly its ultimate distance from the starting point.

[Note that this is *not* a case of *uniform* acceleration or retardation.]

We must first draw the velocity curve; it will be convenient to use squared paper (Fig. 4).

To obtain a good figure we must be careful in our choice of the scales of representation. Distances measured along **OX** represent times; let us take each division of the paper to represent $\frac{1}{3}$ second. Distances measured perpendicular to **OX** represent velocities; let us take each division of the paper to represent 5 ft./sec.

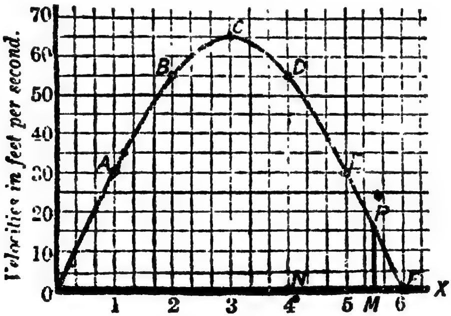

Fig. 4.

We can now plot the points **O, A, B, C, D, E, F** to represent the velocities at the end of 0, 1, 2, 3, etc. . . seconds. Thus **ON** (= 12 horizontal divisions) represents 4 seconds; **ND** (= 11 vertical divisions) represents 55 ft./sec., which is the velocity after 4 seconds. If we now draw by

hand a curve through these points, we may assume that this curve represents fairly accurately the successive velocities of the body. [Thus since **OM** represents $5\frac{1}{2}$ seconds, and **MP** represents 16 ft./sec., the velocity after $5\frac{1}{2}$ seconds is probably 16 ft./sec.]

Hence, by § 41, the area **OBDF** represents the distance travelled.

We may estimate this area fairly accurately by counting the number of squares which it includes. Portions of squares should be counted as whole squares when greater than half a square, and omitted from the counting when less than half a square; they should be counted as halves when they appear to be so.

Counting in this way we estimate the area **OBDF** as equivalent to 144 squares. Now each square represents the distance travelled in $\frac{1}{3}$ second when the velocity is 5 ft./sec., *i.e.* represents a distance of $\frac{5}{3}$ feet; for the horizontal side of the square represents a time of $\frac{1}{3}$ second, and the vertical side a velocity of 5 ft./sec.

Hence the 144 squares represent a distance of $144 \times \frac{5}{3}$ or 240 feet.

NOTE.—The curve starts at **O**, because the inital velocity is zero.

EXAMPLES III.

1. A body starts with a velocity of 1000 feet per minute and loses uniformly $\frac{1}{4}$ of its velocity in each minute. Find how far it will move before it comes to rest.

2. The speed of a railway train increases uniformly for the first three minutes after starting, and during this time it travels 1 mile. What speed (in miles per hour) has it now gained and what distance did it travel in the first two minutes?

3. Find the acceleration of a body which moves from rest through $40\frac{1}{2}$ feet in $4\frac{1}{2}$ seconds.

4. A body describes 504 feet from rest in 12 seconds; find the velocity at the end of that time.

5. A body has described 54 feet from rest, with uniform acceleration, in 3 seconds; how long will it take to move over the next 240 feet?

6. A body moving from rest is observed to pass over 44 feet and 52 feet respectively in two consecutive seconds. Find the acceleration.

7. A body moves over 25 feet during the 3rd second and 55 feet during the 6th second of its motion. Find the whole space passed over in 8 seconds.

8. The acceleration of a body is expressed by 150 when the units of length and time are a yard and a minute respectively. What space will the body describe from rest in 16 seconds?

9. A body starts from rest and moves with uniform acceleration 18 (feet/second^2). Find the time required by it to traverse the first, second, and third feet respectively.

10. A body moves from rest with an acceleration of 16 feet per second per second. How long will it take to travel 8000 feet?

11. A body has an acceleration of 25 feet per second per second. Find the velocity acquired while it travels 20,000 feet from rest.

12. What is the acceleration (supposed uniform) of a body which describes 126 feet in the 4th second of its motion from rest?

13. A body, moving from rest with a uniform acceleration, passes through 10 feet in the first two seconds. How far will it be from the starting point at the end of the 3rd second?

14. Employ the graphical method to find the distance traversed in 10 minutes by a train which starts with a velocity of 20 miles an hour, and has its speed diminished at a uniform rate to 5 miles an hour.

15. A body begins to move with a velocity of 12 feet per second, which increases uniformly at the rate of 14 feet per second in each second. How far will it move in 5 seconds?

16. A particle passes over 560 feet in 10 seconds, and its velocity at the end of that time is 90 feet per second. Find the initial velocity and the acceleration.

17. A particle passes over 10 metres in 10 seconds under a uniform acceleration of 10 C.G.S. units. Find the initial and final velocities of the particle.

18. A body has its velocity uniformly increased from 10 feet per second to 20 feet per second while passing over 50 feet. Find the acceleration.

19. Two bodies uniformly accelerated have their velocities increased from u to v, and from U to V respectively, while passing over the same space. Compare the accelerations and the times of describing the spaces.

20. A body moves over 15 feet in the 1st second of its motion, over 63 feet in the 3rd second, and over 111 feet in the 5th second. Is this consistent with the supposition of uniform acceleration?

21. A train proceeding at the uniform rate of 1200 yards per minute is 150 yards behind another, which is just starting from rest with a uniform acceleration of 1 yard per second per second. In how many seconds will there be a collision?

CHAPTER IV.

GRAVITY.—MOTION OF BODIES FALLING VERTICALLY.

43. When a heavy body is unsupported, it is a matter of everyday observation that it falls to the ground, and with a continual increase of speed. More strictly, it falls towards the centre of the earth, vertically, *i.e.* in the direction of a plumb-line, and with a constant, or uniform, acceleration. This tendency to fall is an illustration of Newton's Law of Universal Gravitation, in virtue of which the Earth's mass attracts every mass placed in its neighbourhood. This attractive force of the Earth is called *gravity*, and the acceleration produced by it is called "the acceleration due to gravity" or "the acceleration of gravity." *It is always denoted by the letter g.*

44. The Acceleration of Gravity is the Same for All Bodies. If we allow a coin and a sheet of paper or a feather to drop freely from rest, both will be accelerated downwards, but the coin will reach the ground quicker than the paper, while a balloon will rise in the air instead of falling. From this it might be supposed that different bodies are differently accelerated by the action of gravity, the coin being more accelerated than the paper. But we must not forget that air itself has weight, and, moreover, a body falling through the air has to set in motion the particles of air which it displaces in its descent. Hence a light body of large size encounters more resistance than a body of smaller size, but of the same weight.

But if different bodies be allowed to fall in a tall jar which has been exhausted of air by means of an air-pump, they will all reach the bottom at the same instant, thus showing that all bodies are equally accelerated by gravity.

45. The same thing can be shown more simply without an air-pump by the following experiment, which should be performed by the student before proceeding further.

Experiment I.—Take a penny or other large coin and cut a round disc of paper slightly smaller than the coin. Lay the paper on the top of the coin, and carefully let the latter drop. Although the paper is uppermost, it will remain on the coin, and both will fall together.

Here the coin, by going in front of the paper, overcomes the resistance of the air, which would otherwise retard the motion of the disc.

46. The Acceleration of Gravity is Uniform.

For we find, by experiment, that the spaces described by a falling body are proportional to the squares of the times of fall, which fact agrees with equation (3) of § 36, which holds for uniformly accelerated motion.

Experiment II.—If a stone be allowed to drop from a height of 4 feet, it will reach the ground in half a second. If it be allowed to fall through 16 feet, it will take 1 second. If dropped through 64 feet, it will take 2 seconds.

Now if f be the acceleration, the formula $s = \frac{1}{2}ft^2$, taken in conjunction with these observations, gives

$$4 = \tfrac{1}{2}f\cdot(\tfrac{1}{2})^2, \quad 16 = \tfrac{1}{2}f\cdot 1^2, \quad 64 = \tfrac{1}{2}f\cdot 2^2,$$

whence $f = 32$, in every case.

Hence we conclude that a falling body descends with a uniform acceleration of about 32 feet per second per second.

Observation.—This experiment is, of course, only a very rough one, because it is very difficult to estimate times with sufficient accuracy.

For more exact experiments it is necessary to use Atwood's machine, to be described in Chap. IX., or Morin's Experiment (Exp. 2, § 321).

47. The Intensity of Gravity.—The above and other experiments show that *the acceleration of an unresisted falling body* is uniform, and, since it is the same for different bodies, its magnitude at any place must be a definite quantity.

This quantity varies slightly in different parts of the Earth. It is least at the Equator, where it amounts to only 32·091

F.P.S., or 978·10 C.G.S. units; and it is greatest at the North and South Poles, where it is estimated to be 33·255 F.P.S., or 983·11 C.G.S. units. It also depends on the altitude; it diminishes slightly when we go up a mountain and diminishes slightly down a mine. At London, at the sea level, $g = 32{\cdot}191$ ft. per sec. per sec. $= 981{\cdot}17$ cm. per sec. per sec.

[N.B.—The above numbers are not to be committed to memory.]

For rough purposes it is usual to take

g = 32 feet per second per second ....(1),

g = 981 centimetres per second per second ....(2).

These numbers must be remembered, as they are constantly required. The more accurate value, $g = 32{\cdot}2$ ft. per sec. per sec., should also be remembered, although it is less often used.

The **vertical** at any place may be defined as the direction in which a body falling freely at that place is accelerated by gravity, or the direction of a plumb-line.

OBSERVATION.—It must be carefully borne in mind that g is an **acceleration**, not a velocity. A body falls to the ground with uniform acceleration but with ever-increasing velocity.

48. Motion from Rest under Gravity.—If we neglect the resistance of the air, a stone or other body dropped from a height will fall freely with a uniform acceleration g, or 32 ft. per sec. per sec. The distance fallen s and acquired velocity v at the end of t sec. will be given in feet and feet per sec. respectively by the formulae

$$v = gt = 32t, \quad s = \tfrac{1}{2}gt^2 = 16t^2,$$

obtained by putting $f = g = 32$ in § 36.

The diagram given in Fig. 5 serves to illustrate the motion. The round dots on the vertical line show the relative positions of the body at intervals of one second, each of the smaller divisions being supposed to represent 16 feet. The velocities at each second also are stated on the diagram.

If the line **AB** be held in a horizontal position, it will represent the motion of a body moving from rest in a horizontal line with acceleration f, if the smaller divisions be taken to represent each $\frac{1}{2}f$ units of length.

$f = g = 32.$

Time elapsed in seconds.	Velocity acquired in feet per second.	Distance fallen measured in feet.
t	$v = f \times t$	$s = \frac{1}{2} f \times t^2$
0	$32 \times 0 = 0$	$\frac{1}{2}.32 \times 0^2 = 0$
1	$32 \times 1 = 32$	$\frac{1}{2}.32 \times 1^2 = 16$
2	$32 \times 2 = 64$	$\frac{1}{2}.32 \times 2^2 = 64$
3	$32 \times 3 = 96$	$\frac{1}{2}.32 \times 3^2 = 144$
4	$32 \times 4 = 128$	$\frac{1}{2}.32 \times 4^2 = 256$
5	$32 \times 5 = 160$	$\frac{1}{2}.32 \times 5^2 = 400$

A

B

UNIFORM ACCELERATION IN A STRAIGHT LINE.

If the line AB be placed horizontally, it illustrates the motion of a body moving with uniform acceleration in a straight line.

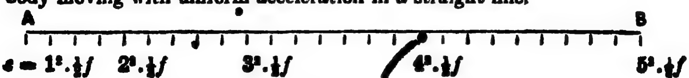

FIG. 5.

MOTION FROM REST UNDER GRAVITY.

49. Distance Fallen in the nth Second.

It will be noticed that the distances traversed in the *individual* seconds are 1, 3, 5, 7, 9, 11 ... times 16 feet respectively; and we should infer that the distance fallen in the nth second is $\frac{1}{2}g\ (2n-1)$ or $16\ (2n-1)$ feet.

This may be shown as in § 39, or as follows :—

The distance fallen in the nth second is the difference of the total distances fallen in n and $n-1$ seconds respectively, and is therefore

$$= \tfrac{1}{2}g.n^2 - \tfrac{1}{2}g\,(n-1)^2$$
$$= \tfrac{1}{2}g\,\{n^2-(n-1)^2\} = \tfrac{1}{2}g\,(2n-1) \quad ....(3).$$

OBSERVATIONS.—It is better not to remember (3), but to obtain it in one or other of the above methods when required.

We notice that in each second the stone falls 32 more feet than in the preceding second. This follows from the fact that in each second the velocity increases by 32 feet per second.

50. If a stone is dropped from a given height h, the time taken in falling and the velocity on striking the ground may easily be got by substituting h for s and g for f in formulae

$$s = \tfrac{1}{2}ft^2, \qquad v^2 = 2fs,$$

which thus become $\frac{1}{2}gt^2 = h, \quad v^2 = 2gh,$

whence $\quad t = \sqrt{\dfrac{2h}{g}}, \qquad v = \sqrt{2gh};$

[Or, if h be measured in feet, and $g = 32$,

$$t = \tfrac{1}{4}\sqrt{h} \text{ seconds}, \qquad v = 8\sqrt{h} \text{ ft. per sec.}]$$

Examples.—(1) To find the depth of a well, when a stone takes $1\frac{1}{2}$ seconds to reach the bottom.

The distance is given by

$$s = \tfrac{1}{2}gt^2 = 16 \times (\tfrac{3}{2})^2 = 36 \text{ feet},$$

which is, therefore, the depth of the well.

(2) If a brick drops off the top of a chimney 100 feet high, in what time will it strike the ground, and with what velocity?

Here we have to find t and v, and we have given $s = 100$, and we know $f\ (= g) = 32$.

Now t is connected with s and f by the equation $s = \frac{1}{2}ft^2$, and so by substituting we get $\quad 100 = 16 \times t^2$;

$$\therefore \quad t^2 = \frac{100}{16}, \text{ or } t = 2\tfrac{1}{2}.$$

Similarly, v is connected with f and s by the formula $v^2 = 2fs$;

$$\therefore \quad v^2 = 2 \times 32 \times 100;$$

$$\therefore \quad v = 8 \times 10 = 80.$$

Therefore the brick will strike the ground in $2\frac{1}{2}$ seconds, with a velocity of 80 feet per second.

51. Bodies Projected Downwards.—If a body be projected downwards with initial velocity u, we merely have to write g for f in the formulae of § 37 to obtain the relations between the time t, final velocity v, and distance fallen s. We thus have

$$v = u + gt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$s = \tfrac{1}{2}(u + v)\,t = ut + \tfrac{1}{2}gt^2 \quad \ldots\ldots\ldots\ldots\ldots(5),$$

$$v^2 = u^2 + 2gs \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Examples.—(1) To find the velocity of projection, if the body descends 2000 feet in 10 seconds.

Let the required velocity be u feet per second. Putting $t = 10$, $s = 2000$, $g = 32$, in the formula

$$s = ut + \tfrac{1}{2}gt^2,$$

we have $$2000 = 10u + 1600;$$

$$\therefore \quad 10u = 400, \quad u = 40.$$

Hence the body must be projected with a velocity of 40 feet per second.

(2) A body is projected downwards with a velocity of 500 centimetres per second; to find (i) the velocity acquired, and (ii) the time elapsed, when it has fallen 50 centimetres.

Let the acquired velocity be v centimetres per second. Then, using the C.G.S. units, we have $s = 50$, $u = 500$, $g = 981$; whence the formula $$v^2 = u^2 + 2gs$$

gives $$v^2 = (500)^2 + 2.981.50 = 250000 + 98100 = 348100;$$

$$\therefore \quad v = 590 \text{ centimetres per second.}$$

The increase of velocity during the interval is therefore 90 centimetres per second, and this increase $= 981t$, where $t =$ time taken in falling;

$$\therefore \quad \text{required time } t = \frac{90}{981} = \frac{10}{109} = \cdot 091 \text{ sec. (approximately).}$$

Or, from $s = \frac{1}{2}(u + v)\,t$, we get $50 = \frac{1}{2}(500 + 590)\,t$, whence t is easily found.

52. Bodies Projected Upwards.—When a body is projected with a given upward velocity u, it is usually convenient to take the *upward* direction as positive. With this convention,

s will always represent the height of the body *above* the point of projection; v will be positive when the body is rising and negative when the body is falling. Since acceleration due to gravity takes place in a downward direction, we must substitute $-g$ for f in our formulae, which now become

$$v = u - gt \quad \text{..............................(7)},$$

$$s = \tfrac{1}{2}(u + v)\,t = ut - \tfrac{1}{2}gt^2 \quad \text{..........(8)},$$

$$v^2 = u^2 - 2gs \quad \text{..............................(9)}.$$

For the upward motion, u is positive, and v, which is equal to u at starting, becomes less and less; for the formula $v = u - gt$ shows that v decreases as t increases, until $gt = u$, when v becomes $= 0$. At this instant the body remains stationary for an instant and then begins its downward course, which (as we shall prove in § 56) occupies exactly the same time as the ascent.

When the body returns to the point of projection, s vanishes; and if the body goes on *below* the point of projection, s is negative.

53. To Find the Time during which the Body Rises.

Since the body ceases rising, and begins falling, when $v = 0$, it follows from the equation $v = u - gt$, that the time of upward motion is given by $0 = u - gt$,

whence $$t = \frac{u}{g} \quad \text{..............................(10)}.$$

54. To Find the Greatest Height to which the Body Rises.

The height is greatest when the body just ceases rising.

We must therefore put $v = 0$ in the equation

$$v^2 = u^2 - 2gs.$$

This gives us s, or h, $= \dfrac{u^2}{2g}$ ....(11).

[This height, $u^2/2g$, is sometimes spoken of as the height *due* to a velocity u. Conversely, if a body falls from a height h, and thus acquires a velocity $\sqrt{(2gh)}$, this is said to be the velocity *due* to the height h.]

Examples.—(1) A stone is thrown upwards with a velocity of 48 feet per second. To find the greatest height, and the time taken in reaching it.

When the stone is at its greatest height, its velocity is zero, and the time is therefore given by

$$0 = 48 - gt = 48 - 32t;$$

$$\therefore \quad t = \tfrac{48}{32} = 1\tfrac{1}{2} \text{ sec.}$$

The height is given by

$$s = u^2/2g = 48^2/8^2 = 6^2 = 36;$$

$$\therefore \quad \text{greatest height} = 36 \text{ feet.}$$

(2) To find the greatest height attained by a body which is thrown vertically upwards with a velocity of 100 ft. per sec.

The velocity at any height s is given by

$$v^2 = u^2 - 2gs = 100^2 - 2.32.s = 10000 - 64.$$

But, when the height is greatest, $v = 0$, and therefore

$$10000 - 64s = 0;$$

$$\therefore \quad \text{greatest height } s = \frac{10000}{64} = 156\tfrac{1}{4} \text{ feet.}$$

55. To Find the Whole Time of Flight.—After reaching its greatest height the body will begin to fall; its height will then decrease, and when this height becomes zero the body will have returned to the point of projection. Hence the time of flight t is found by putting $f = -g$, and $s = 0$, in

$$s = ut + \tfrac{1}{2}ft^2.$$

We therefore have $\quad ut - \tfrac{1}{2}gt^2 = 0;$

$$\therefore \quad t(u - \tfrac{1}{2}gt) = 0;$$

whence, discarding the solution $t = 0$,

we get $\qquad u = \tfrac{1}{2}gt, \quad i.e. \quad t = \dfrac{2u}{g}$ ....(12).

56. Observations.—Comparing (10) and (12), we see that the body rises during half the time of flight. It therefore falls during the other half; hence *the time taken in rising to the highest point is equal to the time taken in returning to the point of projection.*

More generally, the time taken by the body in rising after passing *any* given point is equal to the time taken in again falling to that point; for, as we are not concerned with the

motion *before* first reaching the point, we may treat that point as the point of projection.

From the formula $v^2 = u^2 - 2gh$, by taking the square root, we get $v = + \sqrt{(u^2 - 2gh)}$, which shows that the velocities of passing and repassing any point at a height h above the start are arithmetically the same, though differing in sign. Hence the whole downward flight is an exact reversed copy of the upward flight.

Example.—A ball is thrown up, and caught again in 6 seconds. To find the velocity of the projection, and the greatest height.

Let u be the required velocity. Putting $s = 0$, $t = 6$, $g = 32$, in

$$s = ut - \tfrac{1}{2}gt^2,$$

we have
$$0 = 6u - 16.6^2;$$
whence
$$u = 16 \times 6 = 96 \text{ feet per second,}$$
and
$$\text{greatest height} = \frac{u^2}{2g} = \frac{96 \times 96}{2 \times 32} = 144 \text{ feet.}$$

57. To Find the Time Taken to Reach a Given Height.

First Method.—We may use the equation (8), viz.

$$s = ut - \tfrac{1}{2}gt^2,$$

where s the given height, u the inital velocity, and g the intensity of gravity are supposed known. We want to find the time t; accordingly we must regard the equation as a quadratic in which t is the unknown quantity, and solve it to find t.

Now a quadratic equation has in general two solutions, and these determine the two instants at which the body is at the given height, when it is rising and falling respectively.

Second Method.—Instead of finding the time at once, we may find the velocity v from (9)

$$v^2 = u^2 - 2gs;$$

and we may then find the time from (7)

$$v = u - gt \quad \text{or} \quad t = (u - v) \div g.$$

Examples.—(1) If the velocity of projection is 80 feet per second, the *time of flight* is given by the equation

$$0 = s = ut - \tfrac{1}{2}gt^2 = 80t - \tfrac{1}{2}.32.t^2.$$

Rejecting the factor $t = 0$, this gives

$$t = \tfrac{80}{16} = 5 \text{ sec.}$$

(2) A body is projected upwards with a velocity of 96 feet per second: to find when it will be 80 feet above the point of projection.

We shall employ the above second method; accordingly we have to find the velocity v from the equation

$$v^2 = u^2 - 2gs = 96^2 - 2.32.s.$$

At height 80 feet,

$$v^2 = 96^2 - 64\times 80 = 96^2 - 8^2\times 4^2\times 5 = 32^2\times(9-5) = 32^2\times 4;$$

$$\therefore \quad v = 64, \text{ or } -64,$$

according as the body is rising or falling.

Hence the corresponding times t are given by

$$t = \frac{96-64}{g} = \frac{32}{32} = 1 \text{ second (rising)},$$

or

$$t = \frac{96-(-64)}{g} = \frac{96+64}{g} = \frac{160}{32} = 5 \text{ seconds (falling)}.$$

58. Relative Motion of Two Falling Bodies.—Since the acceleration of gravity is the same for all bodies, *the relative acceleration of two bodies under gravity* (being the difference of their actual accelerations) *is zero.*

Therefore their relative velocity is constant.

This principle is of great use in finding when and where two bodies projected in the same vertical line will meet.

Example.—A stone is dropped from the top of a tower 100 feet high, and at the same instant another stone is projected from the foot with a velocity of 80 feet per second; find when and where they meet.

Initially the velocities of the two stones are 0 and 80; hence the lower one approaches the upper with relative velocity 80 feet per second. And, since both have the same acceleration (viz. that due to gravity), this relative velocity remains constant.

But their original distance apart is 100 feet. Hence they will be together in $\frac{100}{80}$ seconds; that is, in $1\frac{1}{4}$ seconds.

In this time the upper stone will have fallen through a distance

$$s = \tfrac{1}{2}.32.(\tfrac{5}{4})^2 = 25 \text{ feet}.$$

Hence the stones meet 25 feet below the top and 75 feet above the bottom of the tower.

59. Bodies Dropped from a Moving Balloon.—If bodies be let fall from the car of a balloon in motion, they do not start from actual rest but from rest *relative to the balloon. They therefore have initially the same velocity as the balloon.* The same is true when bodies are dropped from a lift or cage which is going up or down a mine.

Example.—A stone is dropped from a balloon at a height of 400 feet above the ground, and it reaches the ground in 6 seconds. To find the velocity with which the balloon was rising.

Let the upward velocity of the balloon be u feet per second. Then the stone starts with an upward velocity u, and in 6 seconds it is at a distance 400 feet below the point of projection. Therefore from

$$s = ut + \tfrac{1}{2}ft^2,$$
$$-400 = u.6 - \tfrac{1}{2}.32.6^2;$$
$$\therefore\ 6u = 576 - 400 = 176;$$
$$\therefore\ u = 29\tfrac{1}{3} \text{ feet per second.}$$

Note that the minus sign is given to g and s in this problem because they are measured *downwards*, and the positive sign to u because it is upward velocity. If the answer had come out negative, it would have indicated a downward initial velocity of the balloon.

59*a*. **Slope and Gradient.**—A curved graph slopes more and less steeply at different points. Thus in Fig. 4, p. 31, the curve slopes more steeply at A than at B, while at C there is no slope. A straight line, however, slopes equally at all points (see the straight graphs in Fig. 5*a*). The slope of a straight line graph could be measured by the angle which it makes with the axis **OX** or with any line parallel to **OX**. It is often more convenient to measure slope by means of the *gradient*.

The gradients of a railway line are usually painted on little posts at intervals along the side of the line for the information of the engine-drivers. If a driver reads that he is travelling up a gradient of 1 in 73, he knows that his engine is rising 1 foot for every 73 feet that it travels.

In Fig. 5*a* **P** is a point on the straight line **AB**, whose ordinate is **MP** and whose abscissa is **OM**. Now suppose **P** to travel along the line from **A** to **B**. As it travels its ordinate and abscissa both increase in magnitude, and it is easy to see from the figure that *whenever the abscissa increases by one inch the ordinate increases by ·4 inch.*

For example, as **P** travels from **H** to **K** its abscissa increases by **FG** or **HL** which is 1 inch, and its ordinate increases by **LK** which is ·4 inch.

It is also evident that if the line sloped more steeply the ordinate would increase more rapidly; and if less steeply, less

rapidly. Hence we see that the *rate of increase of the ordinate* serves to indicate the slope of the line.

Consider next the line **TV** (Fig. 5*a*). If a point travels along this line from left to right, whenever the abscissa increases by one inch the ordinate *decreases* by ·6 inch—*e.g.* in passing from **R** to **S.**

In mathematical language increases and decreases are both described as **increments**, an increase being a positive increment, and a decrease a negative increment.

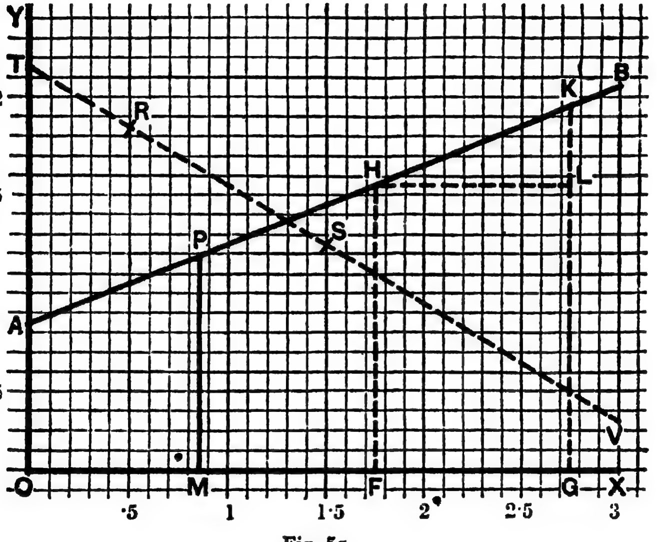

Fig. 5*a*.

Definition.—The **gradient** of a straight line is *the ratio of the increment of the ordinate to the increment of the abscissa*, the increment being found by comparing any two points.

The gradient is positive when the line slopes upward to the right, and negative when it slopes downward to the right. The gradient may be calculated from any two positions on the line.

The gradient of a straight line is the same at all points, since all parts of the line point in the same direction.

The gradient at any point of a curved graph is equal to the gradient of the tangent at that point. We shall assume this without further discussion; but at any rate it is obvious that the portion of the curve near the point of contact roughly coincides with the tangent.

Example.—Calculate the gradient of the line AB in Fig. 5*a*.

If in this figure a point travels from A to B,

increment of ordinate = 1·95″ − ·75″ = 1·2″,

increment of abscissa = 3″ − 0″ = 3″.

Thus gradient = increment of ordinate per unit increment of abscissa = 1·2 ÷ 3 = ·4

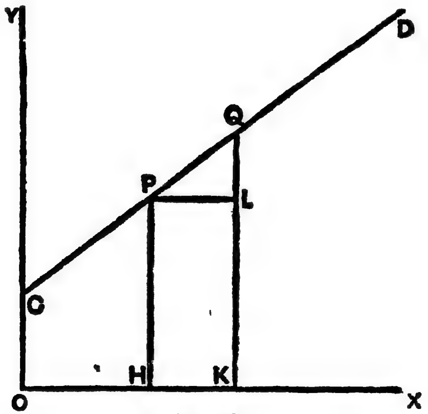

Fig. 5*b*.

59*b*. Applications of Gradients.—(*a*) *The Time-Distance Graph.* Suppose that Fig. 5*b* is a graph representing the motion of a body: and that abscissae (*e.g.* OH, OK) represent times and ordinates (*e.g.* HP, KQ) represent distances travelled during these times. This is called a time-distance graph. (It is not necessarily straight.)

Then comparing the coordinates of P and Q :—

$$\text{increment of abscissa} = \mathsf{OK} - \mathsf{OH} = \mathsf{PL}$$
$$\text{increment of ordinate} = \mathsf{QK} - \mathsf{PH} = \mathsf{LQ},$$

hence $$\text{gradient} = \frac{\mathsf{LQ}}{\mathsf{PL}} = \frac{\text{distance travelled}}{\text{time}} = \text{velocity}.$$

Thus if the time-distance graph be correctly drawn the velocity can be determined from the gradient.

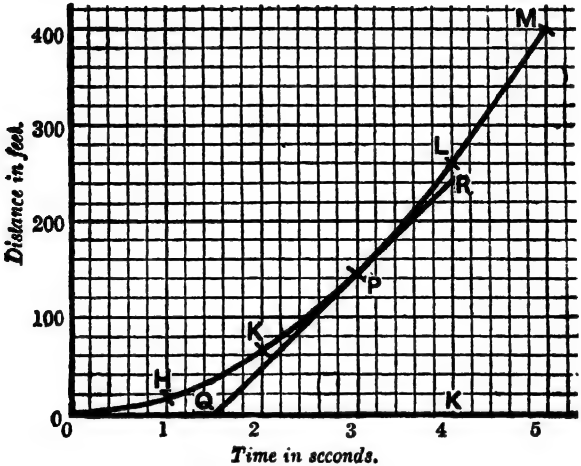

Fig. 5c.

(*b*) *The Time-Velocity Graph.*—Again suppose that the abscissae as before represent times but the ordinates represent velocities. Then as before—

$$\text{increment of abscissa} = \mathsf{PL}$$
$$\text{increment of ordinate} = \mathsf{LQ},$$

hence $$\text{gradient} = \frac{\mathsf{LQ}}{\mathsf{PL}} \quad \frac{\text{velocity}}{\text{time}} = \text{acceleration}.$$

Thus if the time-velocity graph be correctly drawn the acceleration can be determined from the gradient.

Example.—The following table gives the distances (measured from the starting point) through which a stone has fallen at the end of the given times.

Time from the start in seconds	0	1	2	3	4	5
Distance from the start in feet	0	16	64	144	256	400

Determine from a graph the velocity at the end of each second.

The accompanying diagram (Fig. 5*c*) shows the graph.

To determine the velocity at the end of the third second draw the tangent at **P**, viz. **QPR**. Then the velocity at this instant is the same as *the uniform velocity represented by the straight graph* **QR**.

This straight graph indicates a distance represented by **KR**, travelled in a time represented by **QK**.

In the graph **QR** :—

$$\text{increment of ordinate} = \mathbf{KR},$$
$$\text{increment of abscissa} = \mathbf{QK},$$

$$\therefore \quad \text{velocity} = \text{gradient} = \frac{\text{increment of ordinate}}{\text{increment of abscissa}} = \frac{\mathbf{KR}}{\mathbf{QK}}$$
$$= \frac{240}{2\cdot5} = 96 \text{ (ft. per sec.)}.$$

In the same way by drawing tangents at the points **H**, **K**, **L**, **M**, we should find that the velocities at the end of the 1st, 2nd, 4th, and 5th seconds are respectively 32, 64, 128, 160 ft./sec. In practice only approximately accurate results can be expected.

Note.—This process could be continued as follows: We have obtained a set of velocities corresponding to a set of times and could therefore draw a time-velocity graph.

The gradients of this graph would, as shown above, give the accelerations at the corresponding times.

EXAMPLES IV.

In the following examples, the value of g is taken to be 32 feet per second per second, unless the more accurate value, 32·2 feet, is expressly mentioned.

1. Find the distances traversed in feet, and the velocities acquired in feet per second, by a body falling from rest for (i) 5 seconds, (ii) 1 minute, (iii) half an hour, (iv) $\frac{1}{10}$ second.

Obtain the corresponding results in centimetres and centimetres per second, taking $g = 980$.

2. Find the velocities acquired and the times taken in falling freely through (i) 400 feet, (ii) 300 yards, (iii) 3 inches, (iv) 1000 centimetres.

3. Find what space is described in the 2nd, 6th, and 7th seconds respectively by a body falling freely under the action of gravity.

4. A stone, after falling for 2 seconds, strikes a pane of glass, in breaking through which it loses $\frac{1}{4}$ of its velocity. How far will it fall in the next second?

5. A stone dropped over a cliff strikes the ground in 3 seconds. How high is the cliff, and where was the stone when half the time had elapsed?

6. A stone dropped into a well reaches the water with a velocity of 80 feet per second, and the sound of its striking the water is heard $2\frac{7}{12}$ seconds after it is let fall. Find the depth of the well and the velocity of sound in air.

7. A heavy particle is dropped from a given point, and, after it has fallen for $1\frac{1}{2}$ seconds, another particle is dropped from the same point. What is the distance between them when the first has been moving for 4 seconds?

8. A particle, falling freely, passes through 256 feet in the last second of its motion. Find the height from which it fell.

9. A body is thrown upwards with a velocity of 112 feet per second. Determine its velocity at the end of 3 seconds. When will it again have a velocity equal to its initial velocity?

10. A cricket ball thrown up is caught by the thrower in 7 seconds. Draw to scale a figure showing its position at the end of every second from its start.

11. With what velocity must a ball be thrown vertically upwards in order to return to the hand after 4 seconds?

12. An arrow is shot vertically upwards with a velocity of 104 feet per second when it leaves the bow. How long will it be before it reaches the ground again?

13. Find the downward velocity which must be given to a body that it may describe 450 feet in 5 seconds.

14. A body is dropped from a certain height h at the same instant as another is thrown upwards. What initial velocity must the latter have that it may meet the former half way?

15. A ball thrown up is caught by the thrower 7 seconds afterwards. How high did it go, and with what speed was it thrown? How far below its highest point was it 4 seconds after its start?

16. A stone thrown vertically upwards is observed to pass through a point P, and, after an interval of 2 seconds, to pass downwards through the same point. Find the velocity of the stone at P.

17. A ball is thrown vertically upwards with a velocity of 128 feet per second. How long will it take to return to the hand?

18. What velocity must be given to a body projected vertically upwards that it may rise to a height of 900 feet?

19. A body, projected downwards with a velocity of 40 feet per second, describes 216 feet in a certain second. What space did it describe in the previous second?

20. A stone is thrown vertically upward with a velocity of 160 feet per second. How high will it rise, and how long will it be before it returns to your hand?

21. If you let another stone drop down a well at the instant the first is within 20 feet of your hand on its return journey, at what distance below your hand will the two stones meet? (See Ex. 20.)

22. A body is projected upwards with a velocity due to a height $\frac{1}{2}h$ from a point at a height h. Find when it will strike the ground.

23. A stone is thrown vertically upwards with such a velocity as will just raise it to the top of a tower 100 feet high. Two seconds afterwards another stone is thrown from the same place with the same velocity. Determine when and where the stones will meet.

24. A body is projected vertically upwards with a velocity of 80 feet per second. How long will it take to reach the top of a tower 64 feet high?

25. Two stones are dropped from different heights, and reach the ground at the same time, the first from a height of 121 feet, and the second from a height of 100 feet. Find the interval between the instants at which they were dropped.

26. A stone is let fall from a certain point, and, after it has fallen for a second, another stone is let fall from a point 144 feet below that point. In how many seconds will the first stone overtake the second?

27. A body is projected upwards with a velocity of 120 feet per second. Find its velocity when it has reached a height of 25 feet.

28. From a balloon which is ascending with a velocity of 32 feet per second, a ball is let fall, and reaches the ground in 17 seconds. How high was the balloon when the ball was dropped?

29. A balloon has been ascending vertically at a uniform rate for $4\frac{1}{2}$ seconds, and a stone let fall from it reaches the ground in 7 seconds. Find the velocity of the balloon and the height from which the stone is let fall.

30. There are two cages, one of which is in the act of dropping freely under gravity down a pit, while the other is made to descend with uniform velocity. A man in each cage during the descent lets go a stone which he has been holding in his hand. What will be the motion of the stone in each case?

31. Determine by accurate drawing and measurement the gradient of any line which slopes to the axis of x at an angle (i) 24°; (ii) 50°; (iii) − 12°; (iv) − 32°.

32. Construct lines whose gradients are accurately (i) ·2; (ii) 1·3; (iii) − 2; (iv) − ·85. In each case measure the inclination of the line to **OX**.

33. The distances of a train from London at certain times are given in the following table :—

Time : p.m.	3.0	3.20	3.35	3.55	4.25
Distance in miles	75	87	96	108	126

Show graphically that this is consistent with a uniform velocity and determine the velocity from the graph.

34. The leading boat in a race is at the following distances west of a bridge at the times stated :—

Time : p.m.	2.0	2.8	2.14	2.24	2.28
Distance in yards	86	130	163	218	240

Show by means of a graph that this is consistent with uniform velocity, and find this velocity from the graph.

35. A stone thrown up into the air is at the following heights at the times stated :—

After	1 sec.	2 sec.	3 sec.	4 sec.	5 sec.
The height is	64 ft.	96 ft.	96 ft.	64 ft.	0

Draw the time-distance graph and determine the velocities at the end of 2 seconds and 3·5 seconds.

36. A train starting from rest at a station travels the following distances in successive minutes :—100 yd., 300 yd., 500 yd., 700 yd., 900 yd. Draw the time-distance graph, and determine the velocity after 2 minutes and after 4 minutes.

37. A stone is thrown straight down from a high cliff with an initial velocity of 45 ft./sec. The velocities at certain times are given in the following table :—

Time elapsed in seconds =	0,	3,	4,	6,	7.
Velocity in feet per second =	45,	141,	173,	237,	269.

Draw the velocity graph and interpret the gradient.

38. A train travelling at 22 miles an hour comes to the top of an incline. Subsequent velocities are as follows :—

Time elapsed in minutes =	0,	2·5,	3·75,	5·5,	8,
Velocity in miles per hour =	22,	32,	37,	44,	54.

Draw the velocity graph and from it determine the acceleration.

39. Take a point **P** on **OX** 4″ from **O**. With **P** as centre and **PO** as radius describe a quadrant of a circle above **OX**. If this is a velocity graph on which 1″ of abscissa represents 10 seconds and 1″ of ordinate

represents a velocity of 40 ft./sec., determine the acceleration of the moving body after 15 and 35 seconds.

40. A body starts moving from rest and its velocities after 0, 3, 5, 8, 10 seconds are respectively 0, 21, 25, 16, 0 ft./sec. Find its accelerations after 2, 6, and 9 seconds respectively.

EXAMINATION PAPER II.

1. How would you find out whether a body was moving with a uniform acceleration or not?

2. Investigate the formula $s = \frac{1}{2}ft^2$, and deduce a corresponding expression in the case where the particle has an initial velocity u.

3. Find the acceleration necessary to make a body move from rest through 10 feet in 5 seconds.

4. A body begins to move with a velocity of 10 feet per second, and the velocity increases uniformly at the rate of 2 feet per second in each second. Find the space described in 5 seconds, and the velocity of the body when it has moved through 200 feet.

5. Explain a convenient method of representing geometrically the velocity of a body moving according to a fixed law, and the distance passed over by it.

6. If the velocity of a body increase from 12 to 13 feet per second while it moves over a distance of 5 feet, what is the acceleration? Indicate the course of reasoning upon which your calculation is based.

7. What is meant by the statement $g = 32$? What units are employed in this equality?

8. A body is projected vertically upwards. Show that its velocity at any point of its path is the same (save for the *sign*) in descending as in ascending.

9. A body is projected vertically upwards with a velocity of 112 feet per second. How high will it ascend, and what will be its velocity after 5 seconds?

10. A stone is dropped over the edge of a cliff, and half a second afterwards another stone is thrown down with a velocity of 18 feet per second. When will the second stone overtake the first, and what will be the distance between them at the end of 5 seconds after the first stone was dropped?

PART II.

MASS AND FORCE.

CHAPTER V.

NEWTON'S FIRST LAW.

60. **Kinetics.**—Hitherto we have considered motion in the abstract from a *kinematical* point of view. We shall now consider motion generally with reference to (1) what moves, and (2) what causes it to move. That is, we shall have regard to the *mass* of the moving body, and to the *force* which causes or modifies its motion. This portion of the subject is called *kinetics*, in contradistinction to kinematics.

61. **Newton's Three Laws of Motion.**—As has been mentioned in the Introduction, Newton's Axioms or Laws of Motion are accepted as the foundation on which the relations between matter, motion, and force are built up. They are usually translated somewhat as follows :—

FIRST LAW.—**That every body preserves in its state of rest or of uniform motion in a right line, except in so far as it is compelled by forces to change its state.**

SECOND LAW.—**That change of motion is proportional to the impressed motive force, and takes place along the right line in which that force is impressed.**

THIRD LAW.—**That reaction is always opposite and equal to action, or that the actions of two bodies on each other are always equal, and tend in opposite directions.**

We now proceed to examine Newton's Laws in detail.

62. The First Law furnishes us with the following

DEFINITION.—**Force** *is that which tends to change the state of rest or uniform motion of a material body.* (§ 2.)

Force may manifest itself to our senses in various ways. If we *push* or *pull* a body, we exert a force on it; and if the body is acted on by no other force, we shall set it in motion. Again, if we lift a heavy body off the ground, we shall have the body exerting a force on our hand, owing to its *weight*; and when we let the body go, this weight causes it to begin falling. A *magnet* placed near a bar of iron exerts a force of attraction on the iron. All these forces are capable, in suitable circumstances, of setting in motion, or changing the motion of, the bodies on which they act.

63. Evidence in Favour of Newton's First Law.—The fact that a body at rest would, if left to itself, remain at rest, will probably be regarded as an obvious truism. It is not, however, so obvious that a body, if left to itself, would continue to move for ever with uniform velocity in a straight line; for common experience affords us no examples of bodies moving in this manner. The reason is that it is practically impossible to isolate a body from the action of force.

A stone, if projected along a sheet of smooth ice, will continue to skid along for a considerable distance, and will move in a straight line, and the smoother the ice the longer will it travel. If the ice were perfectly smooth, and there were no air to resist the motion, the stone would always continue to travel with uniform velocity. But no ice is perfectly smooth, for even with the smoothest ice there is a small amount of friction. This, together with the resistance of the air, produces a small force on the stone, which gradually stops it, changing its state from a state of motion to a state of rest. When the stone has come to rest, these resisting forces cease to exist, and hence the stone remains at rest.

If a man stands upright in a railway carriage, then, so long as the motion of the train is uniform and in a straight line, he will not feel that he is being *pushed forward* in any way. But if the train suddenly stops, the man will fall forwards owing to his tendency to go on moving.

64. Inertia and Mass.—Newton's First Law is sometimes called the law of inertia. It states that material bodies are unable of their own accord to change their state of rest or motion.

We know that some bodies are much easier to set in motion, or stopped when moving, than others. The efforts required

to produce the same change of velocity in different bodies are proportional to the masses of the bodies.

Mass has been defined in § 5 as "quantity of matter." The property in virtue of which more or less effort is required to change the velocity of a body is sometimes called *inertia*, so that mass may be said to be a measure of inertia.

65. Momentum: DEFINITION.—The **momentum** of a body is a quantity measured by the *product of its mass and its velocity*.*

The momentum of a system of bodies is the sum of the momenta of its different parts. If the mass be doubled, the momentum, with the same velocity, will be doubled, and with double that velocity it will be quadrupled.

If m denotes the mass, and v the velocity of the body,

$$\text{the momentum} = mv \quad \text{..................(1).}$$

The unit of momentum *is the momentum of a unit mass moving with unit velocity.*

In the foot-pound-second system, the unit of velocity is a velocity of one foot per second. Hence the F.P.S. unit of momentum is the momentum of a pound of any substance moving at the rate of one foot per second. This may be written 1 lb. ft./sec.

Example.—The momentum of a 500-pound cannon-ball, when fired with a velocity of 1,000 feet per second, is

$$= 500 \times 1{,}000 = 500{,}000 \text{ foot-pound-second units.}$$

The C.G.S. unit of momentum is the momentum of a mass of one gramme moving with a velocity of one centimetre per second, written 1 gm. cm./sec.

Example.—If a cannon-ball of 10,000 grammes is discharged with a velocity of 50,000 centimetres per second, its momentum

$$= 10{,}000 \times 50{,}000 = 500{,}000{,}000 \text{ C.G.S. units.}$$

* From the Latin *movimentum*, signifying *motion*.

CHAPTER VI.

NEWTON'S SECOND LAW.

66. Newton's Laws as given in § 61 are fairly close translations of the original Latin statements of them. Now, although Newton was marvellously in advance of his time, it is not surprising that these original statements do not fully conform to our modern nomenclature. This is at all events the case for the Second Law, which does not explicitly mention the *mass* of the body acted on, or the *time* of the force's action.

From elementary considerations, it is easy to see that, if we double the mass of the body acted on, we must also double the acting force, in order to produce the same velocity in the same time. And, again, if a force acts upon a body for two seconds, it is natural to suppose that the velocity produced will be twice as great as when the force acts for one second. This seems an obvious inference from the equation $v = ft$, but its proof depends upon observation (as does that of all the laws), and it is amply verified by experiments of all sorts.

Hence we must suppose the "quantity of motion" to be the *momentum*, defined as in § 65, and we may conveniently settle the time difficulty by introducing (as on several previous occasions) the notion of *rate*.

67. The Second Law, accordingly, in its amended form, will be as follows :—

The rate of change of momentum is proportional to the applied force, and the change of momentum takes place in the direction of the straight line in which the force acts.

This is the form most strongly recommended for learning.

[*Another plan*, often adopted, is to interpret the words "impressed motive force" as signifying the product of the force into its time of action. This is now called the impulse of the force. The Law will then read as follows :—

Change of momentum is proportional to the impulse of the force, and takes place in the direction in which the force acts.]

Newton's First Law gave us a definition of force; his Second Law tells us how different forces are to be compared and measured. We are not concerned in this chapter with the cases in which the force is inclined to the direction of motion.

Examples.—(1) If a cricket ball is thrown with a velocity of 50 feet per second, the impressed force used in throwing it is twice as great as if *the same ball* were thrown with a velocity of 25 feet per second, for the impressed forces are proportional to the momenta produced, and are therefore in the ratio 2 : 1.

(2) If two railway trains, of masses 120 and 90 tons, are started together, and one of them acquires a speed of 60 miles an hour in the same time as the other acquires a speed of 40 miles an hour, the forces exerted by the engines of the two trains, being, by Newton's Second Law, proportional to the momenta, are in the ratio of

$$m_1v_1 : m_2v_2, \quad \text{or} \quad 120\times 60 : 90\times 40, \quad \text{or} \quad 7200 : 3600, \quad \text{or} \quad 2 : 1.$$

(3) If the unit of force is that force which, when acting on 1 lb. for 1 second, imparts to it a velocity of 32 feet per second, to find the measure of the force required to impart a velocity of 60 miles an hour to a railway train of 100 tons in 2 minutes.

Let P be the measure of the required force, and take 1 foot, 1 lb., 1 second as units of length, time, and mass.

The velocity acquired by the train in 2 minutes = 88 feet per second, and the mass of the train = 224,000 lb.

Therefore the momentum imparted by P in 2 minutes

$$= 88\times 224{,}000 \text{ F.P.S. units.}$$

Now a force 1, acting for 1 second, produces 32 such units of momentum.

Therefore a force P, acting for 1 second, produces $32P$ units of momentum; and a force P, acting for 120 seconds (or 2 minutes), produces $32\times 120P$ units of momentum.

This must be equal to the momentum just found;

$$\therefore \quad 32\times 120P = 88\times 224{,}000;$$

$$\therefore \quad P = \frac{88\times 224{,}000}{32\times 120} = 5133\tfrac{1}{3} \text{ units of force.}$$

From the Second Law we derive our method of measuring force, as will now be shown.

68. Standard Kinetic Equation.

Let m be the mass of a body, and f the acceleration produced in it by the action of a force whose measure is P. Then, by the assumed fundamental law of motion (§ 67),

P is proportional to rate of change of momentum,

i.e. to rate of change of mv,

i.e. to $m\times$rate of change of v,*

i.e. to mf

(by definition of acceleration).

Hence we may put $P = kmf$, where k is some constant.

Now let the unit of force be so chosen that it may produce in unit mass a unit of acceleration. That is, let P be 1, when m is 1 and f is 1. Our equation shows us that k will also be 1. Hence, *with this convention*, we may put

$$P = mf \quad\text{..............................}(1).$$

This is the fundamental kinetic equation.

69. **The Poundal.**—With a pound as the unit of mass, an acceleration of 1 ft. per sec. per sec. (sometimes written 1 ft./sec.^{2}, or 1 f.s.s.) is the unit of acceleration. Hence the unit of force that *must* be adopted (if we are to use equation (1) with the F.P.S. units) is the force which, acting on one pound of matter, produces in it an acceleration of one foot per second per second. This unit force is called a **poundal**, and is constant all over the world.

In using the equation $P = mf$, we must carefully remember that, in general, P gives the measure of the force in poundals, m gives the measure of the mass in pounds, and f the measure of the acceleration in f.s.s.

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* For let v change in value from U to V. Then mv changes from mU to mV.

But $$mV - mU = m(V - U);$$

$$\therefore \text{ change of } mv = m \times \text{change of } v;$$

and, considering the changes to be those which take place *per unit time*, we see that the same relation connects the *rates* of change of v and mv.

The equation is sometimes expressed in the form :

$$f = \frac{P}{m}, \text{ or acceleration} = \frac{\text{moving force}}{\text{mass moved}} \ldots\ldots\ldots\ldots\ldots (2).$$

This form will often be useful hereafter, expressing as it does the fact that the acceleration varies directly as the force and inversely as the mass moved.

70. Connection between the Poundal and the Weight of a Pound.—We know that, when a body falls freely, it moves with an acceleration g (= 32·2 units); also the force which causes this acceleration is what we call the *weight* of the body. Now a force 1, acting on a mass 1, produces an acceleration 1. Therefore, by the Second Law,

a force g, acting on a mass 1, produces an acceleration g.

But the *weight* of a pound, acting on the pound, produces this acceleration.

Therefore the weight of a pound is a force g, or contains g units of force.

Dividing by g, we see that

$$\text{the unit of force} = \text{wt. of } \frac{1}{g} \text{ lb., or} = \frac{1}{g} \text{ lb. wt.}$$

$$\text{Thus the } \textit{poundal} = \frac{1}{g} \text{ lb. wt.} = \frac{1}{2} \text{ oz. wt., nearly} \ldots\ldots (3).$$

71. C.G.S. Units: Dyne.—When a centimetre, gramme, and second are the units of length, mass, and time, the corresponding unit of force is called a **dyne**. When equation (1) is used in this system, the force will be expressed in dynes, when the mass and acceleration are in C.G.S. units.

$$\text{Also } \textit{one dyne} = \frac{1}{g} \text{gm. weight} = \frac{1}{981} \text{gm. wt.} \ldots\ldots\ldots\ldots\ldots (4).$$

The dyne is a very small force indeed, being only $\frac{1}{445048}$ of the weight of a pound. For this reason forces are often measured in *megadynes*, the megadyne being one million dynes.

72. Hence we have the following

DEFINITIONS.—*The* **dynamical** *or* **absolute unit of force** *is that force which, when applied to a unit of mass for a unit of time, imparts to it a unit velocity.*

The **poundal** *is that force which, when applied to a pound of matter for one second, imparts to it a velocity of one foot per second.*

The **dyne** *is that force which, when applied to a mass of one gramme during one second, imparts to it a velocity of one centimetre per second.*

In distinction to the above methods of measurement, the *Statical Gravitation* measure of a force is the weight that it will support; and the Statical Unit of Force (as nearly always used in Statics) is the weight of one pound or one gramme, or, indeed, any other weight that may be convenient. (§ 89.)

Example.—To express the *poundal* in *dynes*.

By the Chain-rule of Arithmetic:

Let

$$x \text{ dynes} = 1 \text{ poundal},$$
$$32{\cdot}2 \text{ poundals} = 1 \text{ pound (weight)},$$
$$2{\cdot}205 \text{ pounds} = 1 \text{ kilogramme},$$
$$1 \text{ kilogramme} = 1000 \text{ grammes},$$
$$1 \text{ gramme} = 981 \text{ dynes};$$
$$\therefore \quad x = \frac{1000 \times 981}{32{\cdot}2 \times 2{\cdot}205} = \frac{981000}{71{\cdot}001}$$
$$= \frac{981000}{71} = 13800, \text{ nearly.}$$
$$\therefore \quad 1 \text{ poundal} = 13800 \text{ dynes.}$$

Or thus:

A poundal acting on a pound for one second imparts a velocity of 1 foot per second.

But $\quad$ a pound = 453·7 grammes,

and $\quad$ a foot = 30·48 centimetres;

therefore a poundal acting on 453·7 grammes for 1 second imparts a velocity of 30·48 centimetres per second.

Therefore momentum imparted by a poundal in 1 second

$$= 453{\cdot}7 \times 30{\cdot}48 = 13780 \text{ C.G.S. units.}$$

But a dyne imparts 1 C.G.S. unit of momentum in 1 second:

$$\therefore \quad \text{a poundal} = 13780 \text{ dynes, roughly.}$$

73. Later on, we shall see that $W = Mg$ is a special case of the relation $P = mf$, where W signifies the weight of a body of mass M. Applying this to the case of one pound of matter, we see that its weight is g times its mass. Now the mass, or quantity of matter in the body is constant; but the value of g varies with the locality (§ 47.) Hence, the weight also varies with the locality; that is, the weight of a pound of matter attached to a spring balance would give different indications on the balance at different parts of the world (§ 93).

74. Other Forms of the Kinetic Equation.

Multiply both sides of the equation $P = mf$ by t.

Thus $$Pt = mft.$$

If the body was originally at rest, $ft = v$, and we have

$$Pt = mv \qquad\qquad (5).$$

If it originally had a velocity u, $ft = v - u$, and we have

$$Pt = m(v - u) \qquad\qquad (6).$$

In each case, *force × time = momentum produced*,

or *impulse of force = change of momentum.*

These formulae are very useful for working examples.

We see that *impulse = motive effect* of a force.

Examples.—(1) A force of 3 poundals acts on a mass of 4 ounces What is the acceleration produced?

In equation (1), substituting $P = 3$ poundals,

$$m = 4 \text{ oz.} = \tfrac{1}{4} \text{ lb.},$$

we have $$3 = \tfrac{1}{4}f;$$

$$\therefore\ f = 12 \text{ ft. per sec. per sec.}$$

(2) How far can a force of 10 dynes move a kilogramme from rest in a minute?

Let f be the acceleration. Then the equation $P = mf$ or $10 = 1000f$ gives $f = \cdot 01$ cm. per sec. per sec.

And distance traversed from rest in one minute

$$= \tfrac{1}{2}f.t^2 = \tfrac{1}{2}.(\cdot 01)\times 60^2$$

$$= 18 \text{ centimetres.}$$

(3) If a force of 15 poundals act upon a mass of 13 pounds, what velocity will it generate in 8 seconds?

$$Pt = mv;$$

$$\therefore\ 15\times 8 = 13\times v,$$

$$\text{or}\quad v = 9\tfrac{3}{13} \text{ f.s.}$$

(4) What force, acting for 6 seconds on a mass of 12 lb., will change its velocity from 200 to 320 feet per second?

$$\begin{aligned} Pt &= m(v-u), \\ P\times 6 &= 12(320-200), \\ \text{or}\quad P &= 240 \text{ poundals} \\ &= 7\tfrac{1}{2} \text{ lb. weight.} \end{aligned}$$

75. Equal Forces and Equal Masses.—The above law affords the following definitions:—

Equal forces are those which impart to the *same* mass equal velocities in equal intervals of time.

Equal masses are those in which equal forces (*e.g.* those of action and reaction) produce equal velocities in equal intervals of time.

Thus, two projectiles would have equal *masses* if they reached a vertical wall in the same time, when shot successively from the same fixed cross-bow. And two cross-bows would produce equal *forces* if they could project equal masses (or the same mass) equally fast after the same time of action.

NOTE.—*Velocity* is rate of change of *position*.

Acceleration is rate of change of *velocity*.

Force is measured by rate of change of *momentum*, *i.e.* by mass × acceleration.

EXAMPLES V., VI.

1. What is the momentum acquired by
   (i) a mass of 10 oz. after falling for 1 second,
   (ii) a mass of 1 cwt. after falling for $\frac{1}{2}$ minute;
   (iii) a mass of 1 lb. after falling for 15 minutes;
   (iv) a mass of 1 kilogramme after falling for $\frac{1}{10}$ second?

2. What is the momentum acquired by
   (i) a mass of 1 lb. after falling through 160 feet;
   (ii) a mass of $\frac{1}{2}$ oz. after falling through 4800 yards;
   (iii) a mass of 1 ton after falling through 3 inches;
   (iv) a mass of 1 milligramme after falling through 10 metres?

3. A body, whose mass is 1 lb., moves from rest with an acceleration of 10 feet per second per second. Find the momentum acquired in 6 seconds.

4. A body, whose mass is 1 ton, moves from rest with an acceleration of 6 inches per second per second. Find the momentum acquired by the body in moving through 9 feet.

5. What force acting for 5 seconds will produce a velocity of 120 feet per second in a mass of 6 lb.?

6. How long will be required for a force of 100 poundals to give to a mass of 34 lb. a velocity of 20 miles an hour?

7. A mass of 60 lb. has its velocity increased from 30 to 40 feet per second while it passes over 20 yards. Find the force acting upon it.

8. If, while a mass of 10 oz. passes over 10 feet, its velocity increases from 5 feet per second to 15 feet per second, find the moving force.

9. A constant force, acting on a mass of 16 lb. for 5 seconds, produces a velocity of 20 feet per second. Find the force.

10. A force of 32 poundals acts on a mass of 16 lb. Find the time taken to describe 100 feet from rest, and the space described in the 5th second of motion.

11. A mass m is acted on by a force P, and the mass n by a force Q. Compare the accelerations produced.

12. Find the force (supposed constant) producing the acceleration of a mass of 10 lb., which moves from rest through 120 feet in 5 seconds.

13. What force must be applied for one-tenth of a second to a mass of 10 tons in order to produce in it a velocity of 3840 feet per minute. What would be the momentum of the mass so moving?

[N.B.—A numerical answer is meaningless unless the unit intended is also stated.]

14. A body resting on a smooth horizontal table is acted on by a horizontal force of 8 poundals, and moves on the table over a distance of 10 feet in 2 seconds. Find the mass of the body.

15. A railway train whose mass is 100 tons, moving at the rate of a mile a minute, is brought to rest in 10 seconds by the action of a uniform force. Find the magnitude of the force, and how far the train runs during the time for which the force is applied.

16. A particle, whose mass is m lb., moves from rest under the action of a force of P poundals, which is constant in direction. How far will the particle move in n seconds, and what space will it describe in the nth second?

17. What force, acting for 10 seconds on a mass of 8 lb., will increase its velocity from 200 feet per second to 360 feet per second?

18. A certain force, acting on a mass of 13 lb. for 3 seconds, gives it a velocity of 6 feet per second. How long will it take an equal force, acting on a mass of 1 lb., to move it through a distance of 117 feet from rest?

19. A blow is given to a mass of 2 lb. which causes it to start off with a velocity of 50 feet per second. What is the measure of the impulse?

20. A cannon shot of 1000 lb. strikes a target directly with a velocity of 1500 feet per second, and comes to rest. Find the measure of the impulse.

21. If the cannon shot (in the last question) rebounded with a velocity of 200 feet per second, what would be the measure of the impulse?

22. A mass of 1 kilogramme is acted on by a force of 10 dynes. How far will it move from rest in half-a-minute?

23. A mass of 1 kilogramme starts from rest and acquires a velocity of 1 metre per second after moving over a distance of 1 metre. Find the force acting on the mass.

24. How far will a force of 1 dyne move a mass of 1 kilogramme from rest in 1 hour?

25. A force P, acting on a mass of 50 grammes, increases its velocity in every second by 10 centimetres per second; another force Q, acting on a mass of 1 kilogramme, increases its velocity in every second by 1 metre per minute. Compare the two forces.

26. What force, acting for 5 seconds on a mass of 100 grammes, will increase its velocity from 110 to 600 centimetres per second?

27. If 1 kilogramme, 1 metre, and 1 minute be the units of mass, length, and time, find the dynamical unit of force.

28. If 100 lb., 1 yard, and 1 minute be the units of mass, length, and time, find the dynamical unit of force and the unit of momentum.

29. A uniform force acting on a mass of 6 oz. for 2 seconds generates a velocity of 10 feet per second; find the measure of the force in dynes. [1 foot = 30·5 cm.; 1 lb. = 453·6 gm.]

30. A locomotive draws a load of 200 tons. Find the pull needed (1) at a constant speed, if the friction is ·05 of the load; (2) if the friction is the same, and the speed rises from 30 feet/second to 60 feet/second in 1 minute. [$g = 32$ feet/second^2.]

EXAMINATION PAPER III.

1. Enunciate Newton's Laws of Motion.

2. Define *momentum*. What momentum is produced when a mass of 10 lb. falls freely for 10 seconds?

3. Explain the equation $P = Mf$.

4. A particle, whose mass is 1 lb., is projected along a rough horizontal plane and comes to rest with uniform retardation in 4 seconds, at a distance of 64 feet from the starting point. Find the retarding force.

5. Find the acceleration produced, and the momentum acquired in half-a-minute, when a force of

(i) 32 poundals acts on a mass of 1 ton;
(ii) 50 dynes acts on a mass of 1 gramme.

6. A 30-ton mass is moving on smooth level rails at 20 miles an hour; what steady force can stop it (*a*) in half-a-minute, (*b*) in half-a-mile? Specify the force completely.

7. A mass of 16 lb., initially at rest on a smooth horizontal plane, is acted on by a uniform horizontal force for 5 seconds, and during the next 5 seconds the mass moves 375 feet along the plane. Find the magnitude of the force in pounds weight.

8. A mass of 1 lb. is moving at the rate of 30 feet per second, and 1 minute later it is moving at the same rate, but in the opposite direction. What force must have acted on the mass during the interval?

9. If a force equal to the weight of 10 grammes pulls a weight of 1 kilogramme along a smooth level surface, find the velocity when the body has moved 218 metres.

10. An ounce, a second, and an inch being taken as the units of mass, time, and length, respectively, compare the (dynamical) unit of force with the weight of a pound.

CHAPTER VII.

NEWTON'S THIRD LAW.

76. Newton's Third Law may be stated thus:

To every action there is an equal and opposite reaction;

Or, Action and reaction are always equal and opposite.

Here **action** means the force which one body exerts on another, and the law states that the second body always exerts on the first an equal force in the opposite direction in the same straight line. This force is called the **reaction** of the second body on the first. It is important to notice that the law is true whether the bodies are at rest or in motion, and whether they press against one another through being in contact, or act on each other at a distance (as a magnet acts on a bar of iron), *provided they act* **directly** *on one another*, *i.e.* not through a third intermediate body, nor through a system of such bodies or machines.

77. Statical Illustrations of Action and Reaction.

(1) "*If anyone presses a stone with his finger, his finger is also pressed by the stone.*"

(2) "*If a horse draws a block of stone tied by a rope, the horse is, so to speak, drawn back equally towards the stone.*"

Of course this reaction of the stone does not actually make the horse move backwards towards the stone, but only *tends* to do so; or, what is more correct, tends to prevent the horse from moving forward under the action exerted by his feet on the ground. If the rope were suddenly cut, and the horse continued to exert the same effort with his feet as before, he would start so quickly into motion that he would probably

fall over forwards. As Newton puts it, the pull of the rope "impedes the progress of the one by the same amount that it promotes the progress of the other."

(3) If a ladder is allowed to lean against a wall, the ladder presses against the wall and the wall pushes with an equal force against the ladder. The action of the ladder tends to overturn the wall, and will actually overturn it if the masonry is weak and gives way. The reaction of the wall on the ladder prevents the ladder from falling over, as it would at once do if it were placed in the same position without such support.

78. Different Kinds of Action and Reaction.

(i) *Thrust:* DEFINITION.—When the action and reaction of two bodies tend to keep them apart from one another, or to prevent them from moving towards one another, they constitute what is called a **thrust**, or a **push**, or "force of pressure."

(ii) *Pull:* DEFINITION.—When the action and reaction of two bodies tend to keep them together or to prevent them from separating, they constitute a **pull**, or **tension**.

(iii) *Attraction and repulsion:* DEFINITION.—When bodies act on one another *at a distance* (as a magnet acts on a bar of iron), the force between them is called an **attraction** if it tends to bring them together, or a **repulsion** if it tends to separate them.

Thus the Earth's *attraction* causes bodies to fall to the ground with the acceleration g (Chap. IV.).

(iv) *Friction:* DEFINITION.—When the action and reaction between two bodies tend to prevent them from sliding one along the other, they constitute what is known as **friction**.

79. Applications of the Third Law to Locomotion.

(1) The act of *walking* affords an excellent example of the equality of action and reaction, as well as of the properties of friction. In starting off to walk we press backwards on the ground with our feet, and the reaction of the ground gives us an equal and opposite impulse forwards, which sets us in motion.

This action and reaction are due to friction. If we try to walk across a smooth sheet of ice, we shall experience some difficulty, because only a very small amount of friction can be called into play between our feet and the ice.

(2) *Motion of a horse and cart.*—When a horse and cart are just starting into motion the horse exerts a forward pull on the cart, and this pull sets the cart in motion.

If follows from Newton's Third Law that the cart exerts an equal and opposite backward pull on the horse. If this were the only force acting on the horse, the horse would move backwards towards the cart instead of forwards, and this we know is not the case.

But the *action* of the horse's feet in the act of walking presses backwards on the ground, and therefore the equal and opposite *reaction* of the ground (due to friction) tends to push the horse forwards. This reaction exceeds the backward drag of the cart by an amount sufficient to produce the acceleration with which the horse starts into motion.

80. The Principle of Conservation of Momentum.

In the last chapter we saw that forces might be measured by the rates of change of momentum they produce (or tend to produce); hence the Third Law asserts that the rates of change of momentum of two bodies due to their action and reaction are always equal and opposite.

The momentum of one body will increase at the same rate that the momentum of the other decreases. This will always be algebraically true, provided that we make the same conventions with regard to sign for momenta as for velocities (*i.e.* we regard the momenta of bodies moving in one direction as positive, and the momenta of bodies moving in the other direction as negative). The only effect of the action and reaction will be to transfer momentum from one body to the other, without altering the algebraic sum of their momenta. In other words:

The total momentum of a system of moving bodies in any direction is not altered by the mutual reactions of the several bodies.

This property is called the **Principle of Conservation of Momentum.** It holds good for any number of bodies.

81. **The recoil of a gun affords a good illustration of this principle.** The explosion of the powder inside the barrel exerts equal and opposite impulses on the shot and the gun, and causes them to move in opposite directions with equal momenta. Hence, if the speed of the shot be given, the speed of recoil can be found.

Example.—If a 700-lb. shot be fired from a 75-ton gun, with a speed of 1200 feet per second, to find the speed of recoil of the gun.

Here the momentum of the gun is equal and opposite to that of the shot.

Now, momentum of shot $= 700 \times 1200$ foot-pound-second units, and

$\therefore$ momentum of gun is also $= 840{,}000$ F.P.S. units.

But mass of gun $= 75 \times 2240 = 168000$ lb.;

$$\therefore \quad \text{velocity of recoil} = \frac{\text{momentum}}{\text{mass}} = \frac{840000}{168000} = 5 \text{ feet per second.}$$

82. Let M, m be the masses of the gun and shot, V, v their velocities.

Then, since the momenta are equal and opposite,

$$\therefore \quad MV = -mv \quad \text{...........................(1).}$$

Neglecting for a moment the negative sign, this gives us the proportion $V : v :: m : M$: or, the velocities of the gun and shot are inversely proportional to their masses.

Again, if the velocities of two bodies (measured in the same direction) are changed from U, u to V, v by a collision between them, and if the masses of the bodies are M, m, then, since the changes of momentum are equal and opposite,

$$M(V - U) = -m(v - u),$$

or

$$MV + mv = MU + mu \quad \text{...............(2),}$$

or, momentum after the action = momentum before the action.

For an experimental verification of this equation see § 323.

Example.—A ball of mass 3 lb., moving with velocity 2 feet per second, is struck by a ball of mass 1 lb., moving in the same direction with a velocity of 10 feet per second. If after the blow the smaller ball comes to rest, find the subsequent velocity of the larger one.

Momentum of smaller ball before blow	$= 1 \times 10 = 10$ units,
" " " after "	$= 0$;
$\therefore$ change of momentum of smaller ball	$= 0 - 10 = -10$ units.

The change of momentum of the larger ball is equal and opposite;
and therefore $= +10$ units.

But, before the blow, momentum of larger ball $= 3 \times 2 = 6$ units;

$\therefore$ after " " " " $= 6 + 10 = 16$ units;

and its mass $= 3$ lb.;

$\therefore$ its velocity $= \frac{16}{3} = 5\frac{1}{3}$ ft. per sec.

Or, working by the formula, $MV + mv = MU + mu$,

$$3 . V + 1 . 0 = 3 . 2 + 1 . 10,$$

$$3V = 6 + 10, \quad \therefore \quad V = 5\tfrac{1}{3}.$$

83. **Measurement of Mass.**—The Second Law has furnished us with a means of measuring *force* (Chap. VI.). The Third Law affords a means of measuring *mass*. For, since the changes of momentum of two bodies due to their action and reaction are equal, the changes of velocity produced in opposite directions are inversely proportional to the masses of the bodies.

This property follows from (2) by writing it in the form

$$\frac{M}{m} = \frac{v - u}{U - V}.$$

As the velocities can be observed, the relation between the masses may be thence deduced.

Example.—A truck of 6 tons, travelling at 3 miles an hour, collides with another truck at rest, and both move on together at 2 miles an hour. To find the mass of the second truck.

Taking a mile, an hour, and a ton as units of length, time, and mass, the momentum of the first truck is decreased by the collison by $6 \times (3 - 2)$ or 6 units, and therefore 6 units of momentum are imparted to the second truck.

But the velocity acquired by the latter is 2 units;

hence its mass $= \frac{6}{2} = 3$ units of mass;

$\therefore$ the mass of the second truck is 3 tons.

Or, as we should more commonly express it, the second truck *weighs* 3 tons.

84. **Reaction of Motions Relative to the Earth.**—Newton's Third Law shows that when a man jumps off the ground, he communicates to the Earth an amount of momentum equal and opposite to that of his own motion. But the mass of the Earth is so great—being about 6,067 million billion tons—

that the *velocity* thus imparted to the Earth is absolutely imperceptible, and it is destroyed directly the man comes down again.

Moreover, the Earth yields slightly under the man, so that, instead of the motion getting transmitted to the whole Earth, only a slight vibration is produced in the Earth in his immediate neighbourhood. In the case of a man jumping, the vibration is imperceptible; but larger moving masses, such as traction engines and railway trains, as also sudden explosions, often shake the ground for a considerable distance.

85. **Tension.**—Another important extension of the Third Law is the assumption that the tension in a flexible inelastic and weightless string is the same throughout its length, even when it passes round any number of smooth pegs or pulleys (§ 120).

A surface is *smooth* when it can exert no frictional force on bodies in contact with it (§ 78).

So also the *thrust* (§ 78) is the same at all points of a straight rigid rod.

EXAMPLES VII.

1. A shot weighing 16 lb. is fired from a gun weighing 4 tons with a velocity of 1120 feet per second. Find the velocity with which the gun recoils.

2. A ¾-ton shot is discharged from an 81-ton gun with a velocity of 1620 feet per second. What will be the velocity with which the gun will recoil, if the mass of the powder be neglected?

3. A shot weighing 1 gramme is fired from a gun weighing 1000 grammes. If the velocity of the shot be 400 metres per second, find the velocity of recoil.

4. A gun weighing 5 tons is charged with a shot weighing 28 lb. If the gun be free to move, with what velocity will it recoil when the ball leaves it with a velocity of 100 feet per second?

5. A shot of 700 lb. is fired with a velocity of 1120 feet per second from a 35-ton gun. Find the velocity with which the gun recoils. If the recoil of the gun be resisted by a steady force of 1,960,000 poundals, through what space will it recoil?

6. A shot weighing 896 lb. is fired with a velocity of 2000 feet per second from a 100-ton gun. If the gun recoil through a distance of 8 feet, find the average force in poundals resisting the recoil.

7. A shot, whose mass is 700 lb., is moving with a velocity of 1000 feet per second, and, entering the side of a stationary ship whose mass is 8000 tons, it remains embedded in it. Find the velocity which it communicates to the ship.

8. A ball whose mass is 3 lb. is moving with a velocity of 60 feet per second, and impinges directly on another ball whose mass is 9 lb., and which is moving in the same direction with a velocity of 20 feet per second. The two balls move together after impact. Find their common velocity.

9. A body impinges on another of twice its mass at rest. If the two move together after impact, show that the impinging body loses two-thirds of its velocity by the impact.

10. A shell at rest bursts into two parts, the smaller of which is one-fourth of the whole. Find the ratio of the initial velocities of the parts.

11. A shell, which is moving with a velocity of 60 feet per second, bursts, in the line of its motion, into two parts, whose masses are respectively 36 lb. and 6 lb. The velocity of the larger piece is increased in the direction of motion by 15 feet per second. Find the velocity of the smaller.

12. A body A, of weight 5 lb., strikes a body B at rest, weighing 200 lb., with a velocity of 100 feet per second. Find the velocity of B, supposing A to be brought to rest by the impact.

13. Two balls, whose masses are 5 grammes and 20 grammes, and whose velocities are 200 centimetres and 25 centimetres per second, respectively, approach each other in opposite directions, and, after impact, move on together. Find their common velocity.

14. Two inelastic balls, weighing 15 lb. and 30 lb., impinge directly on each other in opposite directions. If the velocity of the smaller be 12 feet per second, find the velocity of the larger in order that the motion may be entirely destroyed by the impact.

15. Three spheres, whose masses are respectively 5, 7, and 13 lb., lie in the same straight line. The first impinges on the second with a velocity of 20 feet per second, and, after impact, the two move on together, and impinge, without rebounding, on the third. Find the final velocity of the three spheres.

CHAPTER VIII.

WEIGHT AND ITS RELATION TO MASS.

86. Weight: DEFINITION.—*The* **weight** *of a body is the force with which it is attracted towards the Earth.*

When we lift a body off the ground, we have to exert a certain force in order to overcome its weight. If the body rests on a table, it presses on the table with a force equal to its weight. If the body is unsupported, it will fall to the ground; hence Newton's First Law of Motion shows that some force must be acting on it. This force is the body's weight. We shall now show that

87. The Weights of Different Bodies are Proportional to their Masses.

As we have already remarked in §§ 70 and 72, the force which causes the acceleration of gravity in a body is called its weight, so that the kinetic equation $P = mf$ becomes in this case $W = mg$ ....(1).

And we have seen in § 44 that g is constant for all bodies at the same place.

Hence $\dfrac{W}{m}$ is constant, *i.e.* the weight of a body is proportional to its mass.

88. To Express the Weight of a Given Mass in Dynamical Units of Force.

The formula $W = mg$ may be expressed in words as follows:—

$$\text{weight (in dynamical units)} = (\text{mass}) \times g.$$

Thus **weight of a pound = 32 poundals,**

and **weight of a gramme = 981 dynes.**

When the weight of a body is measured in dynamical units of force, we speak of it as the "absolute weight" of the body. Hence *absolute weight of mass* $M = Mg$, or, in words,

$$(\text{absolute weight}) = (\text{mass}) \times (\text{acceleration of gravity}).$$

Conversely, as stated before in §§ 70, 71,

$$\text{a force of one poundal} = \tfrac{1}{32} \text{ weight of one pound}$$
$$= \text{weight of half-an-ounce.}$$

And $\quad$ a force of 1 dyne $= \frac{1}{981}$ weight of a gramme.

89. **The Gravitation Unit of Force** *is the weight of the unit of mass.*

Where masses are measured in pounds, the gravitation unit of force is the weight of one pound. Where the gramme is taken as the unit of mass, the gravitation unit of force is the weight of one gramme.

The measure of a force in gravitation units is really the measure of the *mass* whose weight is equal to that force.

By "a force of 1 lb." is meant "*a force equal to the weight of* a pound."

Similarly, "a force of 10 tons" and "a force of 5 oz." denote forces equal to the weight of 10 tons or 5 oz., respectively. To avoid confusion, however, it is better always to add the word "weight" or its abbreviation "wt.," and we may therefore speak of the above forces as "1 *lb. wt.*," "10 *tons wt.*," "5 *oz. wt.*" respectively.

In like manner, *forces equal to the weights of* 1 gramme, or 5 kilogrammes, are more briefly spoken of as "1 *gm. wt.*," "5 *kilog. wt.*," respectively.

Whatever be the system of units adopted, we always have

the gravitation unit of force $= g$ absolute units,

the absolute unit of force $= 1/g$ gravitation units.

90. If we follow Newton in merely asserting that forces are *proportional* to the momenta produced, we can use any (the same) units in comparing two different forces. Thus the

formula
$$\frac{P_1}{P_2} = \frac{M_1 f_1}{M_2 f_2}$$

(of which the meaning will be obvious) will occasionally be useful, the forces P_1, P_2, being reckoned in any, the same, weight unit, as pound, ton, etc.

But when we use the Kinetic Equation, and thereby assume the force *equal* to the rate of change of momentum produced, we must keep to the absolute units. In fact, this is the general rule. If the force, as given in any question, is in gravitation measure, we should at once turn it into absolute measure (*e.g.* replace 5 lb. wt. by 160 poundals*), only at the very end of our work going back, if necessary, to the gravitation measure.

The student should spare no pains in becoming familar with the dynamical and gravitational units of force, as well as the difference between " mass " and " absolute weight."

To understand these ideas fully may take some time, but the time will be well spent if this is done before proceeding further. And in working problems the only safeguard against confusion is to specify at each step of the work the units in terms of which the different quantities are measured—a caution which has already been given, but which applies with especial force to problems of the present class.

Examples.—(1) If a bucket of water, weighing 20 lb., is pulled up from a well with an acceleration of 8 feet per second per second, to find, in lb. weight, the force which must be applied to the rope.

Here the weight of the bucket

$$= 20 \text{ lb. wt.} = 20 \times 32 = 640 \text{ poundals.}$$

The force applied to the rope must not only support the weight of the bucket, but must also produce an upward acceleration of 8 F.P.S. units.

Now the force required to support weight of bucket = 640 poundals, and the force required to produce an acceleration 8 in 20 lb. = 8×20

$$= 160 \text{ poundals.}$$

$$\begin{aligned} \therefore \text{ total upward pull on bucket} &= 640 + 160 \text{ poundals} \\ &= 800 \text{ poundals} = 800/32 \text{ lb. wt.} \\ &= 25 \text{ lb. wt.} \end{aligned}$$

Therefore the rope must be pulled with a force equal to the weight of 25 pounds.

Or thus :

Let P poundals be the force applied, acting vertically upwards; $20g$ poundals is the weight, acting vertically downwards.

---

* We may occasionally, in our work, invent such a term as a *ton-al* for the gth part of a ton wt. (or we may call it 1 ton-ft./sec.^{2}).

Resultant, or *net*, upward force is $P-20g$;

$$\therefore \quad P-20g=mf=20\times 8,$$
$$P=160+640=800 \text{ poundals.}$$

(2) A force equal to the weight of 5 lb., acting on a body, produces an acceleration of 9600 yards per minute per minute. What is the mass of the body?

Here $\qquad P=$ wt. of 5 lb $=5\times 32$ poundals,

$$f=9600 \text{ yd. per min. per min.} = 8 \text{ ft. per sec. per sec.};$$
$$\therefore \quad 5\times 32 = m\times 8;$$
$$\therefore \quad m=20 \text{ lb.}$$

(3) A man weighing 170 lb. is descending in a lift with acceleration $\frac{1}{5}g$. To find his force of pressure on the floor of the lift.

Let R be the required force. Then the forces acting on the man are his weight $170g$ poundals downwards, and the reaction of the floor equal and opposite to R upwards.

$\therefore$ force producing motion $=170g-R$ poundals downwards.

Calling this P, and putting $m=170$ lb., $f=\frac{1}{5}g$ in $P=mf$, we have

$$170g-R=170\times\tfrac{1}{5}g;$$
$$\therefore \quad R=(170-34)\,g=136g \text{ poundals};$$
$$\therefore \quad \text{required force } R=136 \text{ lb. wt.}$$

(4) If a railway train of 120 tons is pulled by the engine with a force of 3 tons weight, to find how far it will have to travel to acquire a velocity of 60 miles an hour.

Here a force of 3×32 ton-als moves a mass of 120 tons;

$$\therefore \quad 3\times 32=120f, \text{ or } f=\tfrac{4}{5}.$$

Let s be the required distance in feet. The acquired velocity $=88$ feet per second, hence the formula for accelerated motion

$$v^2=2fs$$

gives $\qquad 88^2=2\times\frac{4}{5}\times s,$

whence $\qquad s=440\times 11 \text{ ft.} = \frac{11}{12}$ of a mile.

(5) If a 2-oz. bullet, travelling 1600 feet per second, penetrates 10 inches into a target, to find in lb. the mean resistance of the target.

Let f be the acceleration of the bullet in feet per second per second: then, by the formula $v^2-u^2=2fs$, we have

$$0^2-1600^2=2\times\tfrac{10}{12}\times f;$$

whence $\qquad f=-1600\times 960=-1536000.$

(The acceleration is of course really a retardation.)

Now $\qquad P=mf=\frac{1}{8}\text{(lb.)}\times 1536000$

$$=192000 \text{ poundals}$$
$$=6000 \text{ lb. wt.}$$

91. How Mass is Found by Weighing.—We are now in a position to explain why weighing a body in a pair of scales determines its mass.

A common balance consists of a beam or lever which can turn about its middle point, and at its ends are suspended the two scale-pans. The body to be weighed is placed in one scale-pan, and suitable weights are placed in the other. Now it will be shown in Statics that if the beam remains balanced in a horizontal position, the body and weights must press with equal forces on the two respective scale-pans. We thus infer that the *weight* of the body is equal to that of the weights employed.

But weight is proportional to mass.

Therefore also the *mass* of the body is equal to the mass of the weights used to balance it. If these weights are known multiples and sub-multiples of a pound, their amount is equal to the number of *lb. wt.* in the *weight* of the body, or the number of *lb.* in its *mass*.

92. Difference between Mass and Weight.—Although the weight of a body may thus be measured by the same number as its mass, it is important to distinguish between the mass and the weight of a body. *Mass always represents a quantity of matter in the body, and does not depend on gravity; while weight always means the force with which a body is attracted to the ground.*

So long as we have to compare the weight of one body with the weight of another body, the distinction between mass and weight is unimportant.

It does not matter, for example, whether we regard the weight of a packet of sugar or tea as measuring its relative heaviness *as compared with that of the pound weight* belonging to our scales or as measuring the quantity of material or mass in it.

But, when weight is measured in dynamical units of force, the distinction is at once apparent, for

$$\begin{aligned}\textbf{weight of mass } m &= m \textbf{ times weight of unit mass}\\ &= m \textbf{ gravitation units of force}\\ &= mg \textbf{ dynamical units of force.}\end{aligned}$$

93. A spring balance is often used for the purpose of weighing. One of the simplest forms is shown in Fig. 6. The scale-pan holding the goods to be weighed is suspended from a spiral spring. The spring is thus extended by the weight, and the greater the load the more is it extended. The required weight is indicated by a pointer, which moves up and down with the scale-pan, along a graduated scale at the side of the spring.

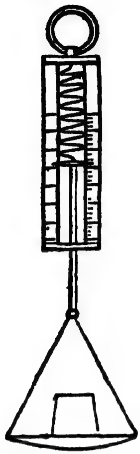

Fig. 6.

Unlike the common balance, the spring balance measures the *absolute weights* and not the *masses* of the bodies placed in the scale-pan. A force of one poundal will always extend the spring by an invariable amount, so that if the scale be graduated in poundals at one place, it will correctly measure forces in poundals at any other place.

Hence a spring balance really gives a constant measure of force.

But it does not give an accurate measure of mass unless it is used to weigh goods at the place for which it is graduated, as the following example will show :—

Example.—Suppose that it is graduated for weighing bodies in pounds at London, where $g = 32 \cdot 191$, and where a pound consequently weighs 32·191 poundals. Then the pointer will always indicate 1 lb. when the scale-pan is pulled down with a force of 32·191 *poundals.* At the Equator a pound only weighs 32·091 poundals, and therefore the weight of a pound mass does not pull the pointer quite down to the graduation marked "1 lb." To bring the pointer down to the 1 lb. reading, we should have to add an extra force of $\frac{1}{10}$ of a poundal, and this would require us to put in about $\frac{1}{20}$ of an ounce more into the scale-pan (since a poundal nearly equals $\frac{1}{2}$ oz. weight). Hence if a tradesman were to buy a spring balance in London, and to use it for weighing goods out at the Equator, he would have to add about $\frac{3}{10}$ per cent. ($\frac{1}{321}$ nearly) to the observed weight to find the weight under the new value of g.

[OBSERVATION.—*Practically,* such differences are too small to be detected except with the most sensitive spring balance.]

94. Apparent Weight of a Man in a Moving Lift.—When a man is ascending or descending in a lift with *uniform* velocity, the reaction of the floor of the lift is exactly equal to the

man's weight. When, however, the lift is being *accelerated* upwards, the reaction of the floor must be greater than the man's weight, because it has not only to support his weight, but also has to give him an upward *acceleration*. And when the lift is being *accelerated* downwards, his weight must exceed the reaction of the floor on his feet by the amount necessary to impart to him the downward *acceleration* of the lift.

The reaction may be found as in Ex. 3, p. 76. The acceleration is evidently upwards when the lift is starting to rise or stopping after falling, downwards when the lift is stopping after rising or starting to fall.

EXAMPLES VIII.

[NOTE.—In the solution of the following examples (as also in some of those on Chapter VI.), there are generally two main processes. First, to find f by means of the Kinetic Equation $P = mf$; second, to put this value of f in the suitable Kinematical Equation (§§ 36, 37). Or, f may sometimes be eliminated between the two equations.]

1. Find the accelerations, the velocities acquired from rest in one minute, and the distance traversed in that minute, by the following masses, when acted on by the given forces, namely:

(i) Mass of 16 lb. under force of 2 lb. weight;

(ii) Mass of 1 cwt. under force of 1 ton weight;

(iii) Mass of 1 kilogramme under force of 1 gramme weight.

2. How far will a mass of 64 lb. be moved from rest by a force of 1 lb. weight in 10 seconds?

3. A force equal to the weight of 1 oz. acts upon a mass of 1 lb. Find the time required to describe from rest a space of 64 feet.

4. If a force equal to the weight of 10 lb. act upon a mass of 10 lb. for 10 seconds, what will be the momentum acquired?

5. A force equal to the weight of 4 lb. acts on a mass of 16 lb. Find the time taken to describe from rest 400 feet, and the space described in the 5th second of motion.

6. A mass of 1 kilogramme is acted on by a force of 10 grammes weight. How far will it move in 20 seconds?

7. A mass originally at rest is acted on by a force which, in 1/384th of a second, gives it a velocity of 5 inches per second. Find the ratio of the force to the weight of the mass.

8. A body resting on a smooth horizontal table is acted on by a force which would statically support 2 oz., and moves on the table over a distance of 10 feet in 2 seconds. Find the mass of the body.

9. While a railway train travels $\frac{1}{4}$ mile on a smooth level line, its speed increases uniformly from 15 to 30 miles an hour. What proportion does the pull of the engine bear to the weight of the train?

10. How far will a force of n kilogrammes weight move a mass of m grammes from rest in t seconds?

11. If a force equal to the weight of 1 gramme pull a mass of 1 kilogramme along a smooth level road, find the velocity when the mass has moved 1 metre from rest.

12. A certain force, acting on a mass of 10 lb. for 5 seconds, produces in it a velocity of 100 feet per second. Compare the force with the weight of a pound, and find the acceleration it would produce if it acted on a mass of 1 ton.

13. A ball, whose mass is 3 lb., is falling at the rate of 100 feet per second. What force expressed in pounds weight will stop it (i) in 2 seconds, (ii) in 2 feet?

14. A body of mass 1 cwt. is moving with a uniform velocity of 10 feet per second, and a force of 1 ton weight begins to act on it. How long will it take this force to increase the velocity to 42 feet per second?

15. A mass of 12 oz., attached to a string, is descending with a uniform acceleration of 12 feet per second per second. Find the force with which the string is stretched.

16. A weight of 2 lb., attached to a string, falls vertically down a mine with uniform acceleration. Find the value of the acceleration if the tension in the string be equal to the weight of 1 oz.

17. A mass of half a ton is supported by a chain, and is moved up and down with a uniform acceleration of 8 feet per second per second. What force is exerted by the chain in each case?

18. A mass of 1 lb. is supported on a plane which is moving downwards with an acceleration of 8 feet per second per second. Find the reaction between the mass and the plane.

19. A man whose weight is 200 lb. is standing in a lift which ascends and descends with uniform acceleration $\frac{1}{8}g$. With what force will he press on the bottom of the lift in each case?

20. If a mass of 32 lb. be placed on a plane which is made to ascend with a uniform acceleration of 2 feet per second per second, find the pressure on the plane.

21. If a railway train of 100 tons is pulled by an engine with a force of 5 tons weight, find how far it will have to travel from rest to acquire a velocity of 30 miles an hour.

22. If a body be acted upon by a force equal to its own weight, and it traverses 176·99 feet during the 6th second of its motion from rest, find the value of g.

23. A cannon ball, whose mass is 2 lb., strikes a target with a velocity of 1000 feet per second, and penetrates 5 feet into the target. Find, in lb. wt., the mean resistance of the target.

24. A mass of 100 kilogrammes is acted on for 1 hour by a force equal to the weight of 1 milligramme. Find the distance traversed from rest.

EXAMINATION PAPER IV.

1. Explain clearly the application of Newton's Third Law of Motion to the case of a horse starting a cart in motion.

2. A 14-lb. shot is fired from a 5-ton gun with a muzzle velocity of 1600 feet per second. Find the velocity of recoil of the gun.

3. Two balls, whose masses are 12 lb. and 8 lb., are moving in the same direction with uniform velocities of 13 and 8 feet per second respectively. When the first ball impinges on the second, they move on together. Find their common velocity.

4. Distinguish between *volume*, *mass*, and *weight*.

5. What is the relation between a poundal and the weight of a pound?

6. If a force equal to the weight of 5 lb. act on a mass of 5 lb. for 20 seconds, what will be the momentum acquired?

7. Find the magnitude of the vertical force which will stop in 4 feet a mass of 5 lb. which has been falling freely for 4 seconds.

8. Explain how it is that the weight of a substance as determined by a pair of scales is the same everywhere, while it will vary if a spring balance be used.

9. A body in falling through 250 feet is retarded by the tension of a string attached to it, so as to occupy 5 seconds in the fall. Find the tension of the string if the mass of the body be 5 lb.

10. A 3-ton cage, descending a shaft with a speed of 9 yards a second, is brought to a stop by a uniform force in the space of 18 feet. What is the tension in the rope while the stoppage is occurring?

CHAPTER IX.

ATWOOD'S MACHINE.

95. The following apparatus was invented about the year 1784. It is now used for illustrating the laws of motion experimentally, and at one time was also employed to determine the intensity of gravity. For the latter purpose, however, it has been superseded by the pendulum.

We could not find g accurately by letting bodies fall down a shot tower or down a mine, and timing them, because the velocity acquired in a few seconds would be so great that the motion could not be timed with sufficient exactness.

96. **Atwood's Machine** consists essentially of a light brass pulley (Figs. 10-12) fixed at a considerable height above the ground, over which passes a fine string supporting two weights P, Q attached to its ends. A *pulley* (Fig. 7) is a wheel with a groove cut round its rim to keep the string which it carries from slipping off. In Atwood's machine it is essential that this wheel should turn very freely, for which reason its shaft usually rolls on sets of supporting wheels called "friction wheels" (Fig. 8), though any other arrangement which answered the same purpose might be used instead.

For measuring the heights of the weights in any position, a scale of inches or centimetres is attached to the pillar or wall on which the pulley is fixed, and for measuring time a clock is provided, whose pendulum ticks every second. In most experiments, the weights P and Q are equal, and a small "rider" R, of the shape shown in Fig. 9, is placed on the top of Q, the string passing through the slot in R. A (Fig. 10) is a platform by which Q can be supported or released at will. S is a ring which is just large enough to let Q pass through,

but which stops the weight R, and C is a fixed platform that will stop the weight Q. Both B and C can be fixed at any desired height, measured by the scale on the pillar.

When the weights Q, R are released, they are together heavier than P, so that they naturally begin to descend, at the same time pulling up the weight P (Fig. 11). When Q reaches the ring B, the weight R is detached, and the equal weights P, Q continue to move on alone (Fig. 12) until Q reaches the platform C, when it also stops. The times taken to fall to the ring B and then to the platform C can be reckoned by the clock, and the scale measures the depths fallen in these intervals.

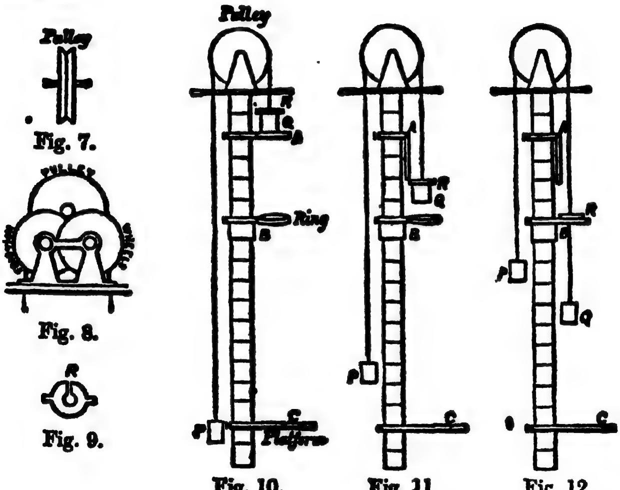

Fig. 7. Fig. 8. Fig. 9. Fig. 10. Fig. 11. Fig. 12.

97. In forming the equations of motion of the two weights in Atwood's machine, it is necessary to make use of the following facts :—

I. *The downward velocity of Q is equal to the upward velocity of P, and the downward acceleration of Q is equal to the upward acceleration of P.*

For, since the length of the string remains constant as one weight falls and the other rises,

$$\therefore \left.\begin{matrix}\text{distance fallen by Q}\\ \text{per unit time}\end{matrix}\right\} = \left\{\begin{matrix}\text{distance risen by P}\\ \text{per unit time};\end{matrix}\right.$$

i.e. downward velocity of Q = *upward velocity of* P.

Hence also

$$\left.\begin{matrix}\text{increase of Q's downward}\\ \text{velocity per unit time}\end{matrix}\right\} = \left\{\begin{matrix}\text{increase of P's upward}\\ \text{velocity per unit time},\end{matrix}\right.$$

i.e. downward accel. of Q = *upward accel. of* P.

II.* *The tension of the string is the same throughout, so that its upward pulls on the weights* P, Q *are equal* (§ 85).

Standard example.—Masses of 3 and 5 lb. hang over a pulley, as in Atwood's machine. Find the tension of the string in lb. weight, and the acceleration of either mass.

Let T be the tension of the string in poundals, and f the acceleration produced. The greater mass will move downwards, and the less upwards. Consider the motion of the 5-lb. mass. The forces acting on it are its own weight, $= 5 \times 32$ poundals, acting downwards, and the tension of the string acting upwards. The total downward force is therefore $160 - T$ poundals, and the mass moved is 5 lb.;

$$\therefore \frac{160 - T}{5} = f \qquad \text{(i)}.$$

Now consider the motion of the 3-lb. mass. The forces acting on it are the tension acting upwards and its weight acting downwards. The total upward force is therefore $T - 96$ poundals, and the mass moved is 3 lb.,

$$\therefore \frac{T - 96}{3} = f \qquad \text{(ii)}.$$

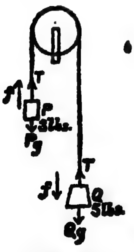

Fig. 13.

Hence, from (i) and (ii),

$$\frac{160 - T}{5} = \frac{T - 96}{3};$$

$$\therefore 8T = 480 + 480 = 960 \text{ poundals};$$

$$\therefore T = \frac{960}{8} \text{ poundals} = 3\tfrac{3}{4} \text{ lb. weight.}$$

Substitute the value of T in poundals in either of the equations (i) or (ii). We thus obtain

$$f = 8 \text{ ft. per sec. per sec.}$$

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* This is only true on the assumption that the pulley is frictionless and of negligible mass.

Second Method.—If the value of T has not to be found, the following (though involving some assumptions) is the readiest method of calculating f:—

Resultant force producing motion

$$= \text{weight of } 5 \text{ lb.} - \text{weight of } 3 \text{ lb.} = \text{weight of } 2 \text{ lb.}$$
$$= 64 \text{ poundals.}$$

Total mass moved is $(3 + 5)$ lb. $=$ 8 lb.;

$$\therefore \quad f = \frac{\text{moving force}}{\text{mass moved}} = \frac{64}{8} = 8 \text{ ft. per sec. per sec.}$$

Example.—(2) Masses of 6 lb. and 8 lb. are hung over a pulley. In what time will they have moved over 7 feet?

Force causing motion

$$= (8-6) \text{ lb. weight} = (8-6)\, 32 \text{ poundals.}$$

Mass moved $= 8 + 6$ lb.

$$\therefore \quad f = \frac{8-6}{8+6} \times 32 \text{ ft. per sec. per sec.}$$
$$= \frac{32}{7} \text{ ft. per sec. per sec.}$$

Let t be the required time; then

$$7 = \tfrac{1}{2} f t^2 = \frac{16}{7} t^2, \quad \therefore \quad t = \sqrt{\frac{49}{16}} = 1\tfrac{3}{4} \text{ sec.}$$

98. Two unequal masses P, Q* ($Q > P$), joined by a string passing over a light pulley, as in Atwood's machine, move under gravity (Fig. 14).

(i) To find their acceleration;

(ii) To find the pull in the string;

(iii) To find the force which the pulley has to support.

(i) The masses being P, Q, the absolute weights of the bodies are Pg and Qg.

Let the pull in the string be T dynamical units of force.

Let the downward acceleration of Q be f, then the upward acceleration of P is also f (§ 97).

The forces acting on Q are therefore Qg downwards and T upwards, and their difference produces the downward acceleration f;

$$\therefore \quad Qf = Qg - T \quad \text{.........................} (1).$$

* For convenience, we now use the letter Q for the total mass of the Q and R before mentioned.

Similarly, from considering the motion of P,

$$Pf = T - Pg \quad \text{..........................(ii).}$$

To find f we must eliminate T (as we do not know the value of T).

By addition, $(Q + P) f = (Q - P) g$;

$$\therefore \quad \text{required acceleration } f = \frac{Q - P}{Q + P} g \quad \text{............(1).}$$

If the weight P is very small, $\frac{Q-P}{Q+P} g$ becomes nearly $\frac{Q}{Q} g, = g$, which is the acceleration of a free body.

Fig. 14.

(ii) To find T, the pull in the string, we must eliminate f from the equations

$$Qf = Qg - T$$

and

$$Pf = T - Pg;$$

and we get

$$P(Qg - T) = Q(T - Pg),$$

or

$$2PQg = (P + Q) T;$$

$$\therefore \quad T = \frac{2PQ}{P+Q} g \text{ dynamical units,}$$

$$= \frac{2PQ}{P+Q} \text{ gravitation units of force ...(2).}$$

Thus the pull in the string is a *harmonic mean* between the weights (see any Algebra). It must always lie *between* them.

(iii) The strings on either side of the pulley pull with a force T. Hence the pulley has to support altogether a force $2T$, equal to the weight of a mass

$$\frac{4PQ}{P+Q}.$$

Notice that this force is less than the sum of the weights, $P + Q$ (by algebra).

99. Alternative Method of Finding the Acceleration.

The weight of Q tends to pull down Q and to pull up P, while the weight of P has the opposite tendency. Hence

the total force tending to accelerate Q downwards and P upwards is the difference of the weights, or $(Q-P)g$ dynamical units. The whole mass accelerated in this way is $Q+P$, and its acceleration is the required acceleration f. Hence the relation

$$\text{mass} \times \text{accel.} = \text{impressed force}$$

gives $$(Q+P)f = (Q-P)g;$$

or, as before $$f = \frac{Q-P}{Q+P}g \quad \text{..........................(1).}$$

100. If the two masses are each equal to M, and a third mass m is placed on the top of one of them, we must write M for P and $M+m$ instead of Q in the expression we have found for f, and we have

$$f = \frac{m}{2M+m}g \quad \text{..........................(3),}$$

which is small if m is small compared with M. This acceleration is the acceleration due to mg, the weight of m, acting on $2M+m$, the total mass of the three weights, as explained above.

101. **Experiments with Atwood's Machine.**

In the preceding article we have discussed theoretically the values of f and T that *ought* to hold, if Newton's Laws are correct. We are now to verify experimentally, by aid of the machine, that the above values *do* hold, and hence we shall infer that Newton's Laws are sound.

Let the equal cylinders each be of mass M, and the *rider* of mass m. Keeping these constant, we can prove that $v \propto t$* and $s \propto t^2$, either of which facts will show that the acceleration produced is constant (or uniform). Let the ring B be fixed at different depths below A, and let the time taken in falling from A to B be noted in each case. It will be found that this time is always proportional to the square root of the

* The symbol $\propto$ denotes "varies as" or "is proportional to." Thus $s \propto t^2$ means that s is proportional to t^2.

depth **AB**, or, what is the same thing, the distance **AB** is always proportional to the square of the time taken. Thus $s \propto t^2$.

To prove that $v \propto t$, we must not only note the times from **A** to the various positions of **B**, but we must also fix the stage **C** at such a depth below **B** in each case that exactly 1 second is taken by the fall from **B** to **C**. The distance **BC** will clearly measure the velocity acquired in the time of going from **A** to **B**. This velocity will be found to vary as the time; in fact, we shall find that $v = \frac{32m}{2M + m}t$, always. Thus we can prove in two ways that *a constant weight acting on a constant mass produces a uniform acceleration.*

Further, if we arrange **B** and **C** so that the times down **AB** and **BC** are the same, we shall find that $\mathbf{AB} = \frac{1}{2}\mathbf{BC}$, or space in time $t = \frac{1}{2}vt$, which is another well-known formula for constant acceleration.

102. Having established the fact that f is constant when P and m are constant, we shall now show how f varies with P and m. For this purpose, we need only observe the length of **AB**, the fall in 1 second, in each experiment. Since $s = \frac{1}{2}ft^2$ and $t = 1$, we see that

$$f = 2s = 2\mathbf{AB}.$$

Let the cylinders be each composed of a number of small detachable sections. By removing an equal number of these from each cylinder, we can keep the moving force P constant, while the mass m varies. By removing one or more from one side, and placing them above the rider on the other, we increase the moving force without altering the whole mass moved.

These experiments prove (within the limits of explainable error) that the acceleration varies directly as P when m is kept constant, and inversely as m when P is kept constant. And, therefore, by algebra, $f \propto P/m$, when both P and m vary.

103. To Find g, the Acceleration of Gravity.

In Atwood's machine we have (§ 100)

$$f = \frac{m}{2M+m} g, \text{ whence } g = \frac{2M+m}{m} f \quad \text{............(3).}$$

The acceleration f may be measured, as in § 102, by observing the time required to fall a given height from rest. The masses M, m may be compared by weighing them in a pair of scales, and, knowing them, g may be found.

For a fuller account of experiments with Atwood's machine see § 322.

104. One Weight Drawn along a Table by Another.

A weight of m lb., hanging freely by a string, draws a weight of M lb. along a perfectly smooth table by means of a string passing over a small pulley at the edge of the table (Fig. 15). *To find the acceleration, and the tension of the string, in lb. wt.*

Let T be the pull of the string in lb. wt.; then its value in poundals $= Tg$.* Also the weight of the hanging mass $= mg$ poundals, and that of the other mass is Mg poundals.

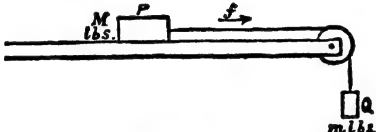

Fig. 15.

Hence, if f be the acceleration of the two masses in feet per second per second, we have, by considering the hanging mass,

$$m.f = mg - Tg;$$

and, by considering the mass on the table,

$$M.f = Tg.$$

Eliminating T, we have

$$(M+m).f = mg;$$

$$\therefore \quad \text{acceleration } f = \frac{m}{M+m} g \text{.........................(4).}$$

* Note that T is not the same thing here as in § 98. The former way, of taking T as the number of poundals, is the better for general work.

Eliminating f, we have

$$mMg - MTg = mTg;$$

$$\therefore \quad \text{tension } T = \frac{Mm}{M+m} \text{lb. wt.} \qquad \ldots\ldots\ldots\ldots (5).$$

The result (4) also follows at once from the fact that the moving force is the m lb. weight hanging freely and the whole mass moved is $m + M$ lb.

Examples.—(1) A mass of 5 lb., on a horizontal table, is connected by a string passing over an edge of the table with a mass of 3 lb. hanging vertically. How far will the latter mass have fallen in one second? (Fig. 15.)

Total mass moved $= (5 + 3)$ lb. $= 8$ lb.

Force producing motion $=$ weight of 3 lb. $= 3 \times 32$ poundals;

$$\therefore \quad \text{acceleration produced} = \frac{3 \times 32}{8} \text{ ft. per sec. per sec.}$$

$$= 12 \text{ ft. per sec. per sec.}$$

$$\text{Distance fallen} = \tfrac{1}{2} ft^2 = \tfrac{1}{2} \times 12 \times 1^2 \text{ ft.} = 6 \text{ ft.}$$

(2) In the preceding example, what is the tension of the string?

Consider the mass on the table. The only force affecting the motion is the tension T of the string;

$\therefore \quad T = mf = 5 \times 12$ poundals $= 60$ poundals $= 1\frac{7}{8}$ lb. weight.

EXAMPLES IX.

1. Two weights are attached to the ends of a string passing over a smooth pulley. Find the acceleration (stating the units employed), the tension in the string (in gravitation units), and the force which the pulley has to support, when the weights are:

(i) 17 lb. and 15 lb.;
(ii) 14 oz. and 2 oz.;
(iii) 3 cwt. and 1 cwt.;
(iv) 15 lb. and 9 lb.;
(v) 3 lb. and 6 lb.;
(vi) 490 grammes and 491 grammes;
(vii) 90 grammes and 19 grammes;
(viii) 4 kilogs. and 5 kilogs.

2. The pairs of weights of Example 1 are laid with one weight resting on a horizontal table and the other hanging vertically over the edge of the table. Find the acceleration and the tension in the string in each of the cases, considering separately the two different arrangements when (a) the lighter, (b) the heavier weight rests on the table.

3. Two masses of 10 and 14 lb., respectively, hang over a smooth pulley. Find the space which will be described from rest in 12 seconds.

4. Two bodies, hanging by a string over a smooth pulley, move with an acceleration of 8 feet per second per second. The mass of the descending body is 3 lb. Find the mass of the other.

5. Two equal weights of $\frac{1}{4}$ lb. each are fastened to the ends of a string which passes over a fixed smooth pulley. What weight must be added to one of these to make them move with an acceleration of 4 feet per second per second?

6. Two bodies, whose masses are 31 and 33 oz., respectively, suspended at the two ends of a thin string passing over a smooth pulley, are allowed to fall freely for 3 seconds. What will be the velocity acquired, and what will be the space traversed by each body?

7. Two weights of 14 oz. and 18 oz. are suspended by a fine thread which passes over a small pulley. If the system be left free to move, find how far the heavier weight will descend in the first 3 seconds of its motion. Find also the tension of the string.

8. Two weights of 15 oz. and 17 oz. are attached to the ends of a fine string which passes over a smooth pulley, and the system is left free to move. In how long will the heavier weight descend through 64 feet?

9. Two weights of 5 lb. and 7 lb., respectively, are fastened to the ends of a cord passing over a smooth pulley supported by a hook. Show that, when they are free to move, the pull on the hook is equal to $11\frac{2}{3}$ lb. weight.

10. Two equal weights of 15 lb. are suspended at the ends of a string which passes over a smooth pulley, and a rider, weighing 2 lb., is attached to one of them so as to start the system in motion. The rider is removed after the system has been in motion for 2 seconds; find how far the weights will move in the next 2 seconds.

11. In an Atwood's machine, the weights attached to the string are 8 lb. and 2 lb., respectively. What weight must be attached to the lighter of the two weights in order that the acceleration may be the same as before, but in the opposite direction?

12. A mass of 2 oz. attached to one end of a string, is descending vertically, the other end of the string being attached to a mass of 6 oz., which slides along a smooth horizontal table. Find the acceleration of the system, and the tension of the string.

13. A mass of 4 lb., attached to one end of a string, is pulled along a smooth horizontal table by another mass, attached to the other end of the string, and hanging over the edge of the table. If the mass on the table moves from rest through 4 feet in 4 seconds, find the mass which hangs over the edge of the table.

14. A mass of 488 grammes is fastened to one end of a cord, which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grammes may rise through a height of 200 centimetres in 10 seconds? (Take $g = 980$.)

15. A cord without weight or friction, passing round a single fixed pulley, has a weight of 500 grammes attached to one end and one of 1000 grammes to the other. When left free, how far will the heavier weight descend in 2 seconds?

16. A weight of 12 oz. is moved from rest on a smooth horizontal table by a weight of 4 oz., which hangs over the edge of the table and is connected with the weight on the table by means of a fine string passing over a small smooth pulley at the edge of the table. Find the tension of the string, the velocity acquired in 4 seconds, and the distance traversed in that time.

17. What must be the masses attached to the ends of the string of an Atwood's machine, in order that the action of a force of 100 poundals on a mass of 1000 lb. may be investigated?

18. It is observed that the larger of the two weights in an Atwood's machine descends through $2\frac{2}{3}$ feet in the first second of its motion from rest. The motion is then stopped, and a weight of 18 lb. is fastened to the smaller weight, which then descends through $5\frac{1}{3}$ feet in the first second. Find the original weights.

EXAMINATION PAPER V.

1. Describe Atwood's machine, and explain how it is used to determine the acceleration of gravity.

2. A couple of unequal weights P and Q hang by a thin flexible cord over a perfectly smooth bar. Find the acceleration, the tension in the cord, the distance travelled by either weight from rest in any given time, and the velocity acquired in the same time.

3. In Atwood's machine, the two weights are 8 lb. and 12 lb., respectively. Find (i) the acceleration, (ii) the distance described by each weight from rest in 5 seconds.

4. Two unequal weights hang by a thin string over a smooth fixed pulley. The heavier of the two is 17 lb. Find the weight of the lighter in order that it may ascend through 4 feet in the first 2 seconds of its motion.

5. Describe an experiment to prove that the weight of half an ounce will produce in a mass of 1 lb. an acceleration of (approximately) 1 foot per second per second.

6. Describe experiments to prove that the distance which a body moving under a given force will traverse in a given time is proportional to the square of the time.

7. Two scale-pans, each weighing 2 oz., are suspended by a weightless string over a smooth pulley. A mass of 10 oz. is placed in one and 4 oz. in the other. Find the tension of the string and the pressure on each scale-pan.

8. A weight of 5 lb. is connected with a weight of 15 lb. by a fine string, and the two weights are so placed that the larger weight rests on a smooth horizontal table, while the smaller hangs over the edge of the table. Find the acceleration produced, and the tension of the string.

9. A weight of 6 lb., connected with another as in the previous question, falls through 16 feet in the first 2 seconds of motion. What is the mass of the weight on the table?

10. What must be the masses attached to the ends of the string of an Atwood's machine, in order that the action of a force of 1000 dynes on a mass of 1000 grammes may be investigated?

CHAPTER X.

WORK, ENERGY, AND POWER.

105. Work.—*A force is said to* **do work** *when its point of application moves in the direction in which the force acts.*

When the point of application moves in a direction opposite to that of the force, **work** *is said to be* **done against** *the force.*

By the "*point of application*" of a force is meant the *particle* on which the force acts. When the former acts, not on a particle, but on a body of any size, the force may be supposed to be applied at some particular point of the body, and the "distance moved by the point of application" means the distance moved by the particle of the body at that point.

Examples of Work.—(1) An engine drawing a train does work, for the train moves in the direction in which the engine pulls. But when the train is being stopped by the brakes, the train does work against the brakes, because the resistance of the latter acts in the opposite direction to that in which the train is moving.

(2) If a heavy body falls to the ground, its weight does work. If we lift it up again, we must do work against its weight.

106. Definition.—*The* **work done by a force** *is measured by the product of the force into the distance through which its point of application moves in the direction of the force.*

In the present chapter, we shall suppose the point of application to be moving in the same straight line as the force. If it is moving in the direction towards which the force tends, the work done will therefore be positive. If it is moving in the reverse direction, we may regard the distance traversed as negative, so that the work done by the force is now a minus quantity.

Hence work done against a force is the same thing as a negative quantity of work done by a force.

107. Definition.—The dynamical or absolute unit of work *is the work done by the dynamical unit of force in moving its point of application through a distance of a unit of length,* whatever system of units be used.

The F.P.S. dynamical unit of work is the **foot-poundal,** and is the *work done by a force of one poundal in moving its point of application through one foot.*

The C.G.S. dynamical unit of work is called the **erg,** and is the *work done by a force of one dyne in moving its point of application through one centimetre.*

Examples.—(1) To find the work done in moving 10 lb. through a distance of 3 feet with an acceleration of 5 feet per second per second.

The force applied to the body

$$= \text{mass} \times \text{accel.} = 10 \times 5 \text{ poundals},$$

and the work done

$$= \text{force} \times \text{distance traversed} = 50 \times 3 = 150 \textit{ foot-poundals.}$$

(2) The work done by a force of 980 dynes in moving through a distance of 10 centimetres is

$$980 \times 10, \quad \text{or} \quad 9800 \textit{ ergs.}$$

(3) To express the foot-poundal in ergs.

By § 72, a poundal contains 13,780 dynes, and a foot contains 30·48 centimetres. Hence, by definition, the foot-poundal is the work done by 13,780 dynes in moving through 30·48 centimetres, and

$$\therefore \quad \textit{a foot-poundal} = 13{,}780 \times 30{\cdot}48, \quad \text{or} \quad 420{,}000 \textit{ ergs.}$$

108. Definition.—**The gravitation unit of work** *is the work done in lifting the weight of a unit mass through a height equal to the unit of length.*

The English gravitation unit is the **foot-pound,** or *the work done in raising one pound of matter vertically through one foot.*

Since the weight of a pound is 32·2 poundals,

$$\therefore \quad \text{a foot-pound} = 32{\cdot}2 \text{ foot-poundals.}$$

The C.G.S. unit is the **gramme-centimetre,** or *the work done in raising one gramme through a height of one centimetre.*

Since a gramme weighs about 981 dynes,

$$\therefore \quad \text{a gramme-centimetre} = 981 \text{ ergs.}$$

Example.—To compare, and express in foot-pounds, the work done by a man weighing 10 stone in climbing a mountain 4,000 feet high; and the work done by the tide between low and high water in raising a ship of 500 tons through 20 feet.

The man raises a weight of 10×14 or 140 lb. through a height of 4,000 feet;

$\therefore$ work done $= 140 \times 4{,}000$ ft.-lb. $= 560{,}000$ ft.-lb.

The tide raises $500 \times 2{,}240$ or 1,120,000 lb. through 20 feet;

$\therefore$ work done $= 1{,}120{,}000 \times 20$ ft.lb. $= 22{,}400{,}000$ ft.-lb.

These are in the ratio of 1 to 40.

109. Energy: Definitions.—By energy is meant *capacity for doing work*.

The only kind of energy that we shall have to deal with is Mechanical Energy, and of this there are two forms, viz. Potential and Kinetic.

The potential energy of a body or system of bodies is *the amount of work which it is capable of performing in virtue of its position* (*or the positions of its parts*).

Examples.—(1) If a million tons of water are stored in a reservoir 500 feet above the sea level, the water may be said to have 500,000,000 *foot-tons of potential energy*, for if the water were allowed to run down to the sea it would be able to perform 500,000,000 foot-tons of work in its descent. By employing the water to drive a series of water wheels in its fall, this work may be utilised for driving machinery.

(2) If, in winding a clock, a weight of 8 lb. is raised to a height of a yard from the bottom of the clock, its *potential energy* is then 24 *foot-pounds*, for in descending again it is able to perform 24 foot-pounds of work. This work is expended in driving the clock, and overcoming the friction of the machinery. When the weight has fallen one foot, its potential energy is only 16 foot-pounds, for it has only 2 more feet to fall, and it has already done 8 foot-pounds of work.

110. If a body of weight W is at a height h above the ground,

$$\text{its potential energy} = Wh \quad\text{..................... (1).}$$

For this is the amount of work its weight would do if the body fell to the ground. It might, for instance, raise another weight W to a height h, by means of a fixed pulley.

Thus, if the mass of a body is M pounds, its weight $= Mg$

poundals, and its potential energy when at a height of h feet above the ground is $= Mh$ ft.lb. $= Mhg$ foot-*poundals* ($g = 32$, or $32{\cdot}2$).

111. Definition.—The **kinetic energy** of a body is *its capacity for doing work in virtue of its motion.* It is measured by the amount of work that the body is capable of performing in coming to rest.

The following illustrations show that a moving body does actually possess energy.

Examples of Kinetic energy.—

(1) A bullet when fired at a wooden target will penetrate a considerable distance into the wood, thereby doing work against the very great resistance to penetration offered by the target. Hence, before the bullet struck the target, it must have possessed kinetic energy, or capacity for doing work.

(2) A stone, when projected vertically upwards, will rise in the air, and thereby do work against gravity. Evidently, the capacity for doing work depends on the initial motion. The original kinetic energy is measured by the work done by the stone against gravity in coming to rest.

Thus, if a mass of 3 pounds is shot upwards with a velocity of 40 feet per second, it will rise to a height h, where (by $u^2 = 2gh$),

$$40^2 = 2.32.h, \quad \text{or} \quad h = 25 \text{ ft.}$$

In rising through this height the body will do 3×25 or 75 foot-pounds of work. Hence the original kinetic energy must have been 75 foot-pounds, or 2400 foot-poundals.

112. To Find an Expression for the Kinetic Energy of a Moving Body.

Suppose a body of mass m to be moving with velocity u, and let us calculate the work it is capable of doing in coming to rest. If the velocity changes from u to 0 under the action of a force of P dynamical units, and if f denote the acceleration, s the space passed over, we have, by § 37,

$$(v^2 - u^2 = 2fs),$$

$$0 - u^2 = 2fs,$$

and, by § 68,

$$P = mf.$$

Hence the work done *by* the force P, moving over a distance s,

$$= Ps = mf.s = 2fs \times \tfrac{1}{2}m = -u^2 \times \tfrac{1}{2}m = -\tfrac{1}{2}mu^2.$$

and the work done *by* the body *against* the force P is equal and opposite to this, and is therefore $= \frac{1}{2}mu^2$.

Therefore the body, in coming to rest, is capable of performing $\frac{1}{2}mu^2$ dynamical units of work, or

The kinetic energy of the body $= \frac{1}{2}mu^2$.....................(2).

Or in words:

The kinetic energy of a body is half the product of its mass into the square of its speed.

When expressed in foot-*pounds*, the kinetic energy

$$= mv^2/2g$$

(or, as some writers put it, $Wv^2/2g$).

Example.—A cannon-ball, of weight 10 lb., is fired horizontally, with a velocity of 1120 feet per second, from a gun, and the weight of the gun, with its carriage, is 5 tons. Find the kinetic energy of the gun immediately after the explosion, expressing it in foot-pounds.

By § 81, we find velocity of recoil to be 1 $f.s.$

Hence kinetic energy $= \frac{1}{2}MV^2 = \frac{1}{2}(5 \times 2240) \times 1^2$

$= 5600$ foot-poundals

$= 175$ ft.-lb.

113. In uniformly accelerated motion the increase of kinetic energy is always equal to the work done by the impressed forces.

We have, in motion under uniform acceleration f,

$$v^2 - u^2 = 2fs;$$

also, if P be the impressed force and m the mass moved,

$$P = mf.$$

$$\therefore \quad Ps = mfs = \tfrac{1}{2}m \times 2fs = \tfrac{1}{2}m(v^2 - u^2),$$

or $$\boldsymbol{Ps = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2} \quad \text{.......................(3)};$$

that is, **work done by P**

= (final kinetic energy) — (initial kinetic energy)

= increase of kinetic energy.

In particular, if the body start from rest, the whole kinetic energy acquired is equal to the work done by the impressed force.

Equation (3) is called the *Equation of Work.*

114. Comparison of the Equations of Momentum and Work.—The student should be careful to distinguish the property just proved from the property which forms the subject of Newton's Second Law [§ 74, equation (6)].

The Second Law states that

$$\begin{aligned} \textbf{force} \times \textbf{time} &= \textbf{impulse} \\ &= \text{change of } \textbf{momentum}; \end{aligned}$$

and the Principle of Work states that

$$\begin{aligned} \textbf{force} \times \textbf{distance} &= \textbf{work} \\ &= \text{change of } \textbf{kinetic energy}. \end{aligned}$$

To calculate the kinetic energy acquired by a body after moving through a given distance under a given force, it is not necessary to find the velocity and substitute in the expression $\frac{1}{2}mv^2$, for the acquired energy is simply the work done by the force.

Examples.—(1) A stone weighing 3 lb. falls through 7 ft. What is its kinetic energy, and what force will stop it in 2 ft.?

$$\begin{aligned} \text{Kinetic energy of stone} &= \text{work done by gravity} \\ &= \text{weight} \times \text{distance fallen} \\ &= 3 \times 7 \text{ ft.-lb.} = 21 \text{ ft.-lb.} \\ &= 21 \times 32 = 672 \text{ ft.-poundals.} \end{aligned}$$

Let P lb. wt. be the force required to stop it in 2 ft. Then we have P lb. wt. acting upwards and the weight 3 lb. wt. acting downwards. Hence the upward force retarding the motion of the stone is $P-3$ lb. wt.

When the stone is brought to rest,

work done *against* retarding force = kinetic energy *lost*;

$$\therefore \quad (P-3) \times 2 \text{ ft.-lb.} = 21 \text{ ft.-lb.};$$

$$\therefore \quad \text{required force } P = 13\tfrac{1}{2} \text{ lb. wt.}$$

(2) A stone weighing 8 oz. falls for 5 sec. What is its momentum, and what force will stop it in 3 sec.?

Weight of stone $= \frac{1}{2}$ lb. wt. $= \frac{1}{2}g$ poundals.

$$\begin{aligned} \text{Momentum acquired} &= \text{impressed force} \times \text{time} \\ &= \tfrac{1}{2}g \times 5 \\ &= 80 \text{ dynamical units (taking } g = 32). \end{aligned}$$

Let P lb. wt. be the force required to stop it in 3 sec. Then the actual retarding force $= (P - \frac{1}{2})$ lb. wt. $= (P - \frac{1}{2})\, g$ poundals, and impulse of this force in 3 sec. = momentum destroyed;

$$\therefore \quad (P - \tfrac{1}{2})\, g \times 3 = \tfrac{1}{2}g \times 5;$$

$$\therefore \quad \text{required force } P = 1\tfrac{1}{3} \text{ lb. wt.}$$

(3) To find the kinetic energy acquired by a kilogramme in falling through a metre.

The weight = 1000 gm. wt. = 1000×981 dynes,

and distance fallen = 100 centimetres.

Hence, by the Principle of Work,

$$\begin{aligned}\text{acquired kinetic energy} &= \text{work done by weight}\\ &= (\text{force}) \times (\text{distance})\\ &= 981{,}000 \times 100 \text{ (dynes, cm.)}\\ &= 98{,}100{,}000 \textit{ ergs.}\end{aligned}$$

115. The Principle of Conservation of Energy.—If a body is started in motion by any force, we see, from § 113, that the kinetic energy acquired is equal to the work done. If after a certain time another force acts on the body, the increase of kinetic energy is equal to the work done by the second force, and the total kinetic energy is therefore equal to the sum of the works done by the two forces. In like manner, if any number of forces act in succession on the body, the final kinetic energy is equal to the sum of the works done by the several forces. If, therefore, the body is ultimately brought to rest, so that its final kinetic energy is zero, it will have done an amount of work equal to that done upon it.

Thus we get as much work out of the body as was previously put into it.

This is a particular case of the **Principle of Conservation of Energy**, which may be briefly stated thus—

Energy can never be created nor destroyed, but can only be transformed from one form into another; or,

The total quantity of energy present in the universe always remains the same.

116. Observations.—The Principle of Conservation of Energy, like Newton's Laws of Motion, does not admit of a perfectly general proof, but is based on evidence derived from experiment. Energy may manifest itself in many other forms besides the ordinary mechanical (kinetic and potential) energy of moving bodies, and it is only when all these forms of energy are taken into account that the principle really holds good.

These forms of energy include energy of vibration which gives rise to sound, heat energy, radiant energy in the form of light, electrical energy, and chemical energy. The tendency of modern physical science is to regard all forms of energy as the kinetic and potential energies of the ultimate molecules of which matter is supposed to be built up. We cannot, of course, tell what these molecules are like or how they really move, for they are far too small to be seen with any microscope. All that we can do is to build up theories of them that will account for physical phenomena. By so doing physicists hope to represent all such phenomena by particular cases of the principles of dynamics.

117. Particular Case of the Principle.

Motion of a Body Projected under Gravity.—If a mass m be projected vertically with velocity u, we have, when the height above the ground is s,

$$v^2 = u^2 - 2gs;$$

$$\therefore \qquad \tfrac{1}{2}mv^2 + mgs = \tfrac{1}{2}mu^2.$$

But mgs is the potential energy (in dynamical units) at height s, and $\frac{1}{2}mv^2$ is the kinetic energy. Hence

kinetic energy + potential energy
= original kinetic energy.

Hence the total energy of the body always remains constant and equal to its original energy.

At the start the energy is wholly kinetic; at the highest point it is wholly potential.

118. Verification for Atwood's Machine.—Let P, Q be the total masses suspended from the ends of a string passing over a pulley, as in Atwood's machine, where $Q > P$. Then, if Q falls through a distance s, P rises through an equal distance s. The work done by the weight of Q is Qgs, and that done against P is Pgs; hence the loss of potential energy

$$= (Q - P)\,gs.$$

If u is the initial velocity, and v the final velocity, the initial and final kinetic energies of the system are

$$\tfrac{1}{2}Qu^2 + \tfrac{1}{2}Pu^2 \quad \text{and} \quad \tfrac{1}{2}Qv^2 + \tfrac{1}{2}Pv^2;$$

hence gain of kinetic energy $= \frac{1}{2}(Q + P)(v^2 - u^2)$.

The Principle of Conservation of Energy requires that

gain of kinetic energy = loss of potential energy,

or that $\frac{1}{2}(Q+P)(v^2-u^2) = (Q-P)gs$,

or that $$v^2 - u^2 = 2\frac{Q-P}{Q+P}gs \quad \text{..................(i).}$$

This relation is satisfied, for in uniformly accelerated motion $$v^2 - u^2 = 2fs,$$
and we have seen in § 98 that
$$f = \frac{Q-P}{Q+P}g,$$
whence (i) follows immediately; hence the sum of the kinetic and potential energies is constant.

Similarly the principle may be verified for the case in which a body is drawn along a smooth horizontal table by a second body falling vertically (§ 104).

119. Applications.—Conversely, we may often determine the motion of a dynamical system by expressing in mathematical language the condition that the total mechanical energy is constant, or that the increase of kinetic energy is equal to the work done on the system.

This is really a most convenient way of finding the acceleration of the masses in Atwood's machine, especially if it is required to find the velocity acquired when these masses have moved through a given distance.

Examples.—(1) If a mass of 1 lb., hanging from the edge of a table, draws a mass of 8 lb. along the table by means of a string, to find the velocity acquired in moving over 1 foot; and the acceleration.

Let the required velocity $= v$ ft. per sec.

Then the total kinetic energy $= \frac{1}{2}(1+8)v^2 = \frac{9}{2}v^2$ ft.-poundals; and the work done by the 1-lb. mass in falling
$$= 1 \text{ ft.-lb.} = 32 \text{ ft.-poundals.}$$

Therefore $\frac{9}{2}v^2 = 32$, or $v^2 = \frac{64}{9}$;

whence $v = \frac{8}{3} = 2\frac{2}{3}$ ft. per sec.

Also the relation $v^2 = 2fs$

gives $\frac{64}{9} = 2f.1$;

whence the acceleration $f = \frac{32}{9}$ ft. per sec. per sec.

(2) A mass of 50 lb. falls from a height of 50 feet, and penetrates 2 feet into loose sand. To find the resistance of the sand in pounds weight.

The kinetic energy acquired in falling is destroyed by the resistance of the sand. Hence the work done on the body by gravity is equal to the work done by the body against the resistance of the sand.

But the body falls altogether 52 feet,

$$\therefore \quad \text{work done by gravity} = 52 \times 50 = 2600 \text{ ft.-lb.};$$

and, since the body moves 2 feet against the resistance,

$$\therefore \quad \text{resistance of sand} = \tfrac{2600}{2} = 1300 \text{ lb. weight.}$$

[Notice that in this example we have not had to calculate the velocity of the body.]

(3) To find the loss of kinetic energy when a mass of 1 lb., moving with a velocity of 10 feet per second, strikes an equal mass of 1 lb., and both continue to move on together.

If v is the common velocity of the two masses after the blow, the constancy of momentum gives

momentum of 2 lb. moving with vel. v = momentum of 1 lb. with vel. 10;

$$\therefore \quad 2v = 1 \times 10, \quad \text{or} \quad v = 5 \text{ ft. per sec.}$$

The kinetic energy of 1 lb. moving with a velocity of 10 feet per second

$$(= \tfrac{1}{2}mv^2) = \tfrac{1}{2}.1.10^2 = 50 \text{ ft.-poundals.}$$

The kinetic energy of 2 lb. moving with a velocity of 5 feet per second

$$= \tfrac{1}{2}.2.5^2 = 25 \text{ ft.-poundals.}$$

Hence the loss of energy $= 50 - 25 = 25$ ft.-poundals $= \frac{25}{32}$ ft.-lb.

120. Tension of a String over a Smooth Pulley.—*When a string passes over a pulley without friction, the tension is the same throughout, if the mass of the string and pulley be neglected.*

Let one end of the string be pulled with a force T, and suppose, if possible, that the pull at the other end is T', and is not equal to T. If a length s of the string is pulled over, the work done on the string by the force T is Ts, and the work done by the string at the other end is $T's$.

Their difference $(T - T')s$ is the mechanical energy communicated to the string and pulley. But there is no friction; hence the system is conservative, and this mechanical energy cannot be lost. Also the string and pulley have no mass; therefore they cannot acquire kinetic energy. Hence the communicated energy $(T - T')s$ must be zero and therefore $T = T'$.

Therefore the pull is the same throughout the string (§ 85).

121. Definitions.—Power *is the rate of doing work.—The* **power** *of an "agent"* (*e.g.* a steam engine, a horse, or whatever does work) *is measured by the amount of work the agent is capable of performing per unit of time.*

The F.P.S. dynamical unit of power is a rate of working of one foot-poundal per second.

The C.G.S. dynamical unit of power is a power of one erg per second.

122. Horse-Power : Gravitation Units of Power.—DEFINITIONS.—The power of a steam-engine is always measured in horse-power.

A horse-power (h.p.) *is a rate of working of*

550 *foot-pounds per second* ....(4)

= 33,000 foot-pounds per minute.

This unit of power was introduced by Watt, who estimated it as being the rate of working of a good horse, and it has been universally adopted by engineers as the unit of power. The power of an engine when expressed in horse-power is spoken of as the *horse-power of the engine.*

[Note that the horse-power is a gravitational unit.]

When engineers speak of *an engine of so many horse-power*—say a 10 horse-power engine—they mean an engine which is capable, under favourable circumstances, of working at 10 horse-power—*i.e.* performing 5500 foot-pounds per second. But such an engine might be worked more slowly and might be used to perform, say, only 4400 foot-pounds per second. It would then be said that the engine was working at $\frac{4}{5}$ of its full horse-power.

Examples.—(1) To find the horse-power of an engine which draws a railway train at 60 miles an hour against a resistance equal to the weight of 1 ton.

Here the engine moves 88 feet per second against a resistance of 2240 lb. wt. Hence it performs

$$88 \times 2240 \text{ ft.-lb. per sec.};$$

$$\therefore \text{ required horse-power of engine} = \frac{88 \times 2240}{550} = 358{\cdot}4.$$

(2) A steam pump raises 11 tons of water 15 feet high every minute. What is its horse-power?

$$\text{Work done per min.} = 11 \times 2240 \times 15 \text{ ft.-lb.};$$

$$\therefore \text{ work done per sec.} = 11 \times 2240 \times 15 \div 60 \text{ ft.-lb.}$$

$$= 11 \times 560 \text{ ft.-lb.}$$

But $\qquad$ one horse-power = 550 ft.-lb. per sec.;

$\therefore$ $\qquad$ required horse-power $= \frac{11 \times 560}{550} = \frac{56}{5} = 11 \cdot 2.$

(3) To express the horse-power in F.P.S. dynamical units.

A horse-power = rate of working of 550 ft.-lb. per sec.

= 550×32 ft.-poundals per sec.

= 17600 F.P.S. dynamical units.

EXAMPLES X.

1. Find the work done by the following forces moving through the given distances :—

(i) a force of 12 poundals through 100 feet;

(ii) a force of 10 lb. weight through 2 inches;

(iii) a force of 10 dynes through 10 metres;

(iv) a force of 5 grammes weight through 2 centimetres.

2. Find the work done in moving the given masses through the given distances with the given accelerations :—

(i) 20 lb. through 4 ft. with an accel. of 2 ft. per sec. per sec.;

(ii) $\frac{1}{2}$ ton through 100 yd. with an accel. of $\frac{1}{28}$ ft. per sec. per sec.;

(iii) 100 gm. through 20 cm. with an accel. of 4 cm. per sec. per sec.;

(iv) 10 kilog. through 10 metres with an accel. of $\frac{1}{10}$ cm. per sec. per sec.

3. If $\frac{1}{2}$ cwt. be raised vertically through 10 feet in each minute, find the work done per second in absolute measure.

4. Find the kinetic energy acquired by the given masses falling freely through the given heights :—

(i) 10 lb. through a height of 100 feet;

(ii) 4 tons through a height of 9 feet;

(iii) 10 grammes through a height of 10 centimetres;

(iv) 1 kilogramme through a height of 10 metres.

5. Find the work done by a fire-engine which discharges every second for a minute 80 lb. of water with a velocity of 50 feet per second.

6. Equal forces act for the same time upon unequal masses M and m; what is the ratio of (i) the momenta generated by the forces, (ii) the amounts of work done by the forces?

7. A cannon ball, whose mass is $\frac{1}{4}$ cwt., falls through a vertical height of 225 feet; what is its energy? With what velocity must such a cannon ball be projected from a cannon to have initially an equal energy?

8. A mass of 10 lb. falls 100 feet, and is then brought to rest by penetrating 1 foot into sand. Find the average pressure of the sand.

9. A body, whose mass is 12 lb., moves from rest with a uniform acceleration of 100 inches per second per second; calculate its velocity, momentum, and energy after it has moved over 20 feet. In what units are your results expressed?

10. A $\frac{1}{2}$-ton shot is discharged from an 81-ton gun with a velocity of 1620 feet per second. Will the gun or the shot be able to do more work before coming to rest, and in what proportion?

11. Distinguish clearly between the *momentum* and the *energy* of a moving body.

12. A stone, whose mass is 25 lb., falls freely under gravity for $\frac{3}{5}$ second. Find its momentum and its kinetic energy at the end of that time.

13. A body, whose mass is 100 grammes, is thrown vertically upwards with a velocity of 980 centimetres per second. What is the energy of the body (i) at the moment of propulsion, (ii) after $\frac{1}{2}$ second, (iii) after 1 second? ($g = 980$.)

14. A shot of 1000 lb., moving at the rate of 1600 feet per second, strikes a fixed target. How far will the shot penetrate the target, exerting upon it an average pressure equal to the weight of 12,000 tons?

15. A steady force applied to a mass of 75 tons, initially moving at the rate of 3 miles an hour, accelerates it at the rate of 4 feet per second per second. Calculate (i) the applied force in pounds weight, (ii) the speed of the body after the lapse of $1\frac{1}{2}$ minutes, (iii) its kinetic energy at the same time, expressing it in foot-tons.

16. A mass of 12 lb., moving along a smooth horizontal plane with a velocity of 60 feet per second, impinges directly on a mass of 18 lb. at rest on the plane. After impact the two masses move on together. Find the loss of kinetic energy.

17. A mass of 5 lb., moving with a velocity of 20 feet per second, impinges directly on a mass of 5 lb. moving with a velocity of 10 feet per second in the same direction, and after impact the two masses move on together. What kinetic energy is lost?

18. Find the average force which will bring to rest, in 20 feet, a cannon ball, whose mass is 1 lb., moving horizontally at the rate of 1600 feet per second. How long will it take to bring it to rest?

19. A shot is fired from a gun, which is fixed, with a certain charge of powder. If the charge of powder be quadrupled, in what proportion will the velocity of the shot be increased?

20. Find, in miles per hour, the speed which would be maintained by an engine of 3 horse-power working against a resistance of 1000 poundals.

21. Determine the horse-power of an engine which will raise 120 lb. of water per minute from a mine 880 feet deep.

22. Find the time which a man weighing 10 stone will take to climb a mountain 3000 feet high, if he can do work at the rate of 4200 foot-pounds per minute.

23. Find the horse-power of an engine which moves at the rate of 45 miles an hour, the weight of the engine and load being 100 tons, and the frictional resistance 20 lb. per ton.

24. A steam crane of 10 horse-power raises a load to a height of 50 feet in $3\frac{1}{2}$ minutes. What is the greatest possible weight of the load?

25. An engine of 64 horse-power draws a load along a horizontal road. The weight of the engine and load is 50 tons, and the frictional resistance 16 lb. per ton. Find the greatest speed which the engine can maintain.

26. How many watts are there in a *force de cheval*?* (Take $g = 981$.)

27. If the units of mass, length, and time be, respectively, 1 lb., 1 yard, and 1 minute, find the dynamical units of energy and power.

28. If the units of mass, length, and time be, respectively, 1 kilogramme 1 metre, and 1 second, find the dynamical unit of energy.

EXAMINATION PAPER VI.

1. Define *work*, *energy*, and *horse-power*. How are they measured?

2. A man weighing 15 stone climbs a mountain $\frac{1}{4}$ mile high. How much work does he do against gravity?

3. A body, whose mass is 5 lb., is thrown vertically upwards with a velocity of 42 feet per second. Find its kinetic energy when it has been in the air 1 second.

4. Show that, when a particle is moved from rest by a constant force acting through a given space, the kinetic energy of the particle is equal to the work done by the force.

5. What is the horse-power of an engine which can project 10,000 lb. of water per minute with a velocity of 80 feet per second, 20 per cent. of the whole work done being wasted by friction?

6. In a railway train the resistance and friction of the rails is 4 lb. per ton. What is the horse-power of an engine which will maintain a speed, on the level, of 45 miles an hour, if the weight of the train be 50 tons?

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* A *watt* = 10,000,000 C.G.S. dynamical units of power, a *force de cheval* = 75 kilogramme-metres per second.

7. A mass of 8 lb., moving with a velocity of 40 feet per second, impinges directly on a mass of 12 lb. moving in the same direction with a velocity of 20 feet per second, and after impact the two masses move on together. Find the loss of kinetic energy.

8. A bullet of mass 1 oz. leaves the muzzle of a gun 3 feet in length with a velocity of 1000 feet per second. Find the average pressure of the powder on the bullet.

9. A body weighing 5 lb. drops through a distance of 100 feet, and is brought to rest by penetrating 10 feet into mud. Find the average resistance of the mud.

10. Enunciate and explain the Principle of Conservation of Energy.

PART III.

THE PARALLELOGRAM LAW.

CHAPTER XI.

COMPOSITION AND RESOLUTION OF VELOCITIES.

123. Representation of Uniform Velocities by Straight Lines.—We shall now deal with motions which are not all in one straight line; and in the first place we shall consider the properties of two or more motions which take place with uniform velocities in different straight lines.

In future, when we speak of a body as "moving uniformly," we shall imply that it is moving *with uniform velocity in a straight line.*

In order to specify completely the velocity of a body, it is necessary to state

(*a*) *How fast* it is moving;
(*b*) *In what direction* it is moving.

The first of these two data is called the **speed** of the body, or the **magnitude** of its velocity, and (if the motion is uniform) is measured by the distance traversed in a unit of time (Chap. I.).

The second is called the **direction** of the velocity, and is the direction of the straight line in which the body moves. It may be specified by referring it to certain fixed directions, such as the vertical and horizontal directions, the points of the compass, etc.

If, then, we draw the straight line which the body actually traverses in a unit of time, the length of this line will measure

the speed, and its direction will indicate the direction of motion; hence the line will be sufficient to completely specify the velocity of the body. Such a line is said to represent the velocity in question.

Thus *uniform velocities may be represented by straight lines.*

Any equal and parallel straight line drawn anywhere would also represent the same velocity, since it would serve equally well to indicate the magnitude and direction. The *line of motion* in Dynamics is of little importance, although the *line of action* in Statics is of great importance.

124. The sense of the direction may be shown by an arrow drawn on or by the side of the line, or by the *order* of the letters used in naming the line. Thus **AB** represents a velocity which in unit time would carry a body from **A** to **B**; **BA** a velocity which in unit time would carry it from **B** to **A** (§ 18).

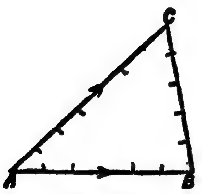

Fig. 16.

Example.—Two boats are sailing, one due east at 6 miles an hour, the other north-east at 7 miles an hour. To represent their velocities in a diagram.

Draw **AB** due east, and on it cut off **AB**, containing six units of length.

Draw **AC**, making an angle 45° with **AB**, and on it cut off **AC**, containing seven units of length.

Then, if a mile and an hour are the units of length and time, **AB**, **AC** represent completely the velocities of the two boats.

125. **Representation of Variable Velocities.**—When a body is not moving in a straight line, its velocity is variable, even if its speed remains constant.

Thus, if the body revolves in a circle so as to describe equal arcs of the circle in equal times, its velocity will be variable.

In dealing with variable velocity, it is usually necessary to specify it by the velocity at *any instant of time.* This is the velocity in a small interval of time, including the given instant, the interval being so short that *neither the speed nor the direction of motion* has time to change in it.

The velocity at any instant is *not* represented by the path *actually* traversed in a unit of time, but by the straight line

which *would* be the path traversed if the velocity were to remain uniform from that instant onwards (as would be the case, by Newton's First Law, if the body were not acted on by any force). This line is a *tangent* to the curve along which the body actually moves. Thus **velocities are always represented by straight lines,** never by arcs of curves.

126. **Relative Velocity.**—As explained in §§ 19-21, the velocity of one body *relative* to another is the velocity with which the first body would *appear* to move if the person observing it were moving with the second body.

In many cases the relative velocity may be found very easily.

Examples.—(1) Two men start simultaneously to walk, one eastwards at 4 miles an hour, the other northwards at 3 miles an hour. To find their relative velocity and the direction in which they separate.

Let the men start from A. Then, in 1 hour the first man will have arrived at B, 4 miles east of A, and the second will have arrived at C, 3 miles north of A.

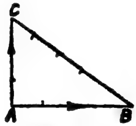

Fig. 17.

Since the two men started together, BC represents the distance the second man appears to have moved away in an hour, as observed by the first.

Therefore BC measures the relative velocity in miles per hour.

Now, by Euclid I. 47, since BAC is a right angle,

$$BC^2 = AB^2 + AC^2 = 4^2 + 3^2 = 16 + 9 = 25;$$
$$\therefore \quad BC = 5.$$

Hence the relative velocity is 5 miles per hour, in a direction parallel to BC.

So, in Fig. 16, the velocity of the second boat relative to the first is represented by BC, and is seen to be 5 (miles per hour).

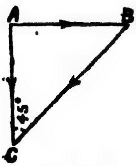

Fig. 18.

(2) A carriage is travelling through a shower of rain, which is falling vertically with a velocity equal to that of the carriage. To show that, to a person in the carriage, the rain appears to fall at an angle of 45° with the vertical, and to find its apparent velocity.

Suppose that at any instant a raindrop appears to coincide with a speck on the carriage-window at A. Then, when the speck (with the carriage) has moved through a horizontal distance AB, the drop will have fallen through an equal vertical distance AC, and the relative positions of the speck

and drop will be B, C. Therefore BC represents the direction in which the drop appears to move away from the speck, *i.e.* the apparent direction of the rain relative to the carriage.

But ABC is a right-angled isosceles triangle, and therefore ACB = 45°;

$$BC = AC\sqrt{2}.$$

Hence the direction of the rain appears to make an angle 45° with the vertical.

Also apparent dist. traversed by drop = $\sqrt{2}\times$ (actual dist. traversed);

$\therefore$ apparent vel. of drop = $\sqrt{2}\times$ (actual vel. of rain).

127. Having given the velocities of two bodies, to construct graphically their relative velocity.

If the velocities of two bodies be represented by two sides AB, AC of a triangle, their relative velocity shall be represented by the third side BC.

Let OP and AC be the paths actually traversed in a unit time by two bodies moving uniformly. We have to find the apparent velocity of the second body to an observer stationed on and moving with the first.

Complete the parallelogram AOPB, and join BC.

Then the *total* change of position of the bodies is the same as if

(i) the first body moved from O to P and the second from A to B,

(ii) the first body then remained at rest while the second moved from B to C.

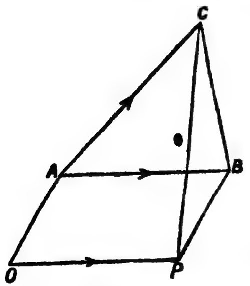

Fig. 19.

Now the first part of the motion does not affect the *relative* position of the second body as seen from the first, for, since PB is equal and parallel to OA, the distance and direction of B when seen from P are the same as those of A when seen from O.

Hence the change per unit time in the relative positions of the bodies is the same as if the first body remained at P, and the second moved from B to C. Therefore BC represents the velocity of the second body relative to the first.

And AB (or OP) and AC evidently represent the velocities of the bodies themselves. Hence the relative velocity is represented by the third side of the triangle ABC.

128. Composition of Velocities.—A body cannot be in two places at the same time, and cannot move in two different ways at the same time, and so cannot have two velocities at the same time. But it is often convenient to consider the motion of a body as made up or *compounded* of several independent velocities.

These are called the **component** velocities of the body, and are to be regarded as *relative velocities* on which the motion of the body depends.

The body's actual velocity is called its **resultant** velocity.

The process of determining the resultant velocity when the components are given is called **compounding** the several velocities.

The definitions of § 23 are perfectly general. But unless the motions are all in one straight line, the resultant velocity is not simply the algebraic sum of the components.

Thus, suppose a river is flowing, a steamer is being driven through the water by its engines, a man is walking across the deck of the steamer, and a fly is crawling up the man's hat. Then the component velocities of the fly are (*a*) the velocity of the water, (*b*) the velocity with which the steamer is driven *relative* to the water, (*c*) the velocity with which the man walks *relative* to the steamer, (*d*) the velocity with which the fly crawls *relative* to the man's hat. Each of these relative velocities affects the motion of the fly, but the actual or *resultant* velocity of the fly is different from any of them.

Examples.—(1) A ship is sailing at the rate of 12 feet per second, and a sailor climbs up the mast at the rate of $3\frac{1}{2}$ feet per second. To find the man's actual velocity.

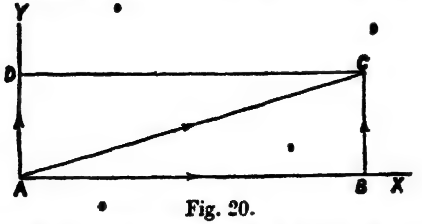

Fig. 20.

Suppose the sailor originally at the foot of the mast at A. Then in one second the motion of the ship carries the foot of the mast from A to B, where AB = 12 feet. But the sailor has climbed up $3\frac{1}{2}$ feet, therefore he is at a point C, $3\frac{1}{2}$ feet above B, and AC is the distance actually traversed in one second.

Now, since **ABC** is a right angle,

$$AC^2 = AB^2 + BC^2 = 12^2 + (\tfrac{7}{2})^2 = 144 + \tfrac{49}{4} = \tfrac{625}{4};$$

$$\therefore \quad AC = \tfrac{25}{2} = 12\tfrac{1}{2};$$

and therefore the sailor's actual velocity is $12\frac{1}{2}$ feet per second.

(2) A man rows a boat through the water at the rate of 3 miles an hour in a direction 60° east of north, in a current flowing southwards at the rate of $1\frac{1}{2}$ miles an hour. To show that the boat will travel due eastwards, and to find its rate of progress.

If a straw, dropped from the boat at **A**, were to drift with the current (supposed constant), it would in an hour reach a point **B**, $1\frac{1}{2}$ miles south of **A**.

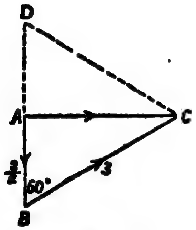

Fig. 21.

But the man has rowed relatively to the water and straw through 3 miles in a direction 60° east of north.

Therefore the boat will have arrived at **C**, where $BC = 3$ miles, and $\angle ABC = 60°$. Complete the equilateral triangle **BCD**.

Then $AB = 1\frac{1}{2}$ miles $= \frac{1}{2}BC = \frac{1}{2}DB$.

Therefore **A** is the middle point of **BD**, and **AC** is at right angles to **AB**.

Therefore the boat's course **AC** is due eastwards.

Also $AC = \sqrt{3}.AB = \frac{3}{2}\sqrt{3}$ miles.

Therefore the boat's actual velocity is $\frac{3}{2}\sqrt{3}$ miles an hour.

Note.—We have, in these examples, tacitly verified the general propositions which we are about to prove. The second example, in particular, may be hard to accept, until the next few articles are read.

129. The Parallelogram of Velocities.—If a body has two component (uniform) velocities represented by the straight lines **AB** and **AD**, and if the parallelogram **ABCD** be completed, the resultant (or actual) velocity of the body will be uniform, and will be represented by the diagonal **AC**.

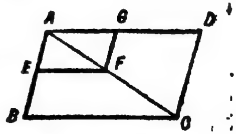

Fig. 22.

Let the body, at the beginning of the second under consideration, be at the point **A**.

From the notion of component velocities, it follows that the position of the body at the end of the second will be the same as if it moved through the space **AB** relative to a line originally placed in the position **AB**, while every point of the line was being moved with a velocity **AD**. The line will thus at the end of the second be

in the position **DC**, while the body will have moved from one extremity to the other of the line, and will therefore be at the point **C**.*

Also, if **AE** and **AG** represent respectively the spaces the body and line would separately describe with the given velocities in any given part of a second, it is evident that the actual position of the body is then at **F**, where **GF** is equal and parallel to **AE**.

Now, since the velocities are uniform,

$$AE/AB = AG/AD.$$

Hence, if the parallelogram **AEFG** be completed, the triangle **AGF** will be similar to **ADC** (Appendix, § 8), and therefore **F** will lie on **AC**, and **AF** will be the same part of **AC** that **AE** is of **AB**, or **AG** of **AD**. Hence the body describes the straight line **AC**, and traverses distances proportional to those traversed by **E** or **G**, and therefore it moves with uniform speed. Hence the resultant velocity is *uniform*, and is represented by the diagonal **AC**.

NOTE.—Some writers infer the *uniformity* of the resultant velocity after the following fashion: Since the component velocities are uniform, or the same at each instant, their resultant is also the same at each instant both in magnitude and direction, that is to say, it is a uniform velocity.

130. Alternative Construction. Since **BC** is equal and parallel to **AD**, it is clear that it will represent the second component velocity just as well as **AD** does. We thus see that *velocities* **AB** *and* **BC** *have a resultant* **AC**.† This leads to a simpler construction for obtaining the resultant of two velocities. Draw a straight line to represent one of them, and from the extremity

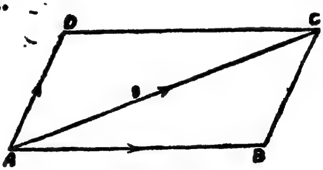

Fig. 23.

* The result, so far, holds good even if the velocities are *not* uniform; but *then* we know nothing about the *path* of the body.

† This important result is frequently referred to as the "Triangle of Velocities."

of this line draw a second line to represent the other; the straight line from the start of the first line to the finish of the second will represent the resultant in magnitude and direction. Care must be taken to draw each line in the right *sense* (§ 124).

Contrast with this the rule given in § 127.

Similarly, if there is a third component, we draw a third line from the end of the second, and the resultant is from the start of the first to the finish of the third line.

When we draw lines in this way, *one way round*, without lifting the pencil from the paper, they are said to be *taken in order*. If arrows be used to denote the *sense*, one arrow will point towards a vertex, while another one points away from it. This construction leads to the following (somewhat important) propositions.

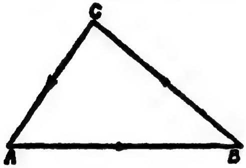

Fig. 24.

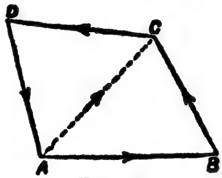

Fig. 25.

131. The Triangle of Velocities.—**If a body have three component velocities which can be represented by the sides of a triangle taken in order, then the body will remain at rest.**

Let the three component velocities be represented by **AB, BC, CA** (Fig. 24).

The resultant of **AB** and **BC** is **AC**, and this compounded with **CA** produces zero velocity, or the body will remain at rest under the three velocities.

To understand how two equal and opposite velocities can be equivalent to a state of rest, suppose an ant to crawl on a book from **A** to **C** while we push the book so that **C** moves to where **A** now is. Or, suppose a passenger to walk, inside a tramcar, with equal speed in the opposite direction to that of the car.

132. The Polygon of Velocities.—**If a body have any number of component velocities, which can be represented by the sides of a closed polygon taken in order, the body remains at rest.**

Let **AB, BC, CD, DA** be the sides of the polygon representing the several component velocities (Fig. 25).

Then the resultant of **AB** and **BC** is **AC**, that of **AC** and **CD** is **AD**, and that of **AD** and **DA** is zero.

Hence the body whose motion is compounded of all the velocities remains at rest at **A**.

133. To find the magnitude of the resultant of two velocities u, v in directions at right angles to one another.

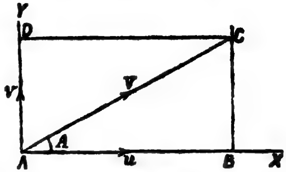

Fig. 26.

Draw **AB**, **AD** at right angles, and let **AB** contain u, and **AD** contain v units of length.

Then **AB**, **AD** represent the two velocities u, v.

Complete the parallelogram **ABCD**.

Then **AC** represents the resultant velocity V.

By Euclid I. 47, $AC^2 = AB^2 + BC^2 = AB^2 + AD^2$;

$$\therefore \quad V^2 = u^2 + v^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1);$$

$\therefore$ **resultant velocity** $V = \sqrt{(u^2 + v^2)}$.

Example.—If the component velocities are 5 and 12 units respectively,

$$V^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2,$$

and resultant velocity $\quad V = 13$ units

134. Resolution of Velocities.—It may happen that we are given the resultant velocity in magnitude and direction, and that we have to find what are the component velocities along two given lines which have the given velocity for their resultant. This process is called resolving the given velocity into components in the given directions, and is the reverse of compounding velocities.

The only two cases which are required for the solution of elementary problems are those in which the given directions are at right angles to each other, and the direction of the given velocity makes angles of either 45° and 45° or 30° and 60° with them.

135. CASE I.—*A velocity V is inclined at angles of 45° and 45° to two given perpendicular lines. To resolve it into components along these lines.*

Let AC represent the velocity V, and let AB, AD be the two given directions. Draw CB parallel to DA, and CD to BA; then, by the Parallelogram of Velocities, AB and AD or BC represent the required components.

Since $\angle BAC = 45°$ and $\angle ACB = 45°$, therefore ABC is a right-angled isosceles triangle; [Appendix, § 5]

$$\therefore \quad AB = BC = \frac{AC}{\sqrt{2}}$$

$$= \tfrac{1}{2}\sqrt{2}\,.\,AC.$$

Hence the required components are $\frac{1}{2}\sqrt{2}.V$ along AB and $\frac{1}{2}\sqrt{2}.V$ along AD.

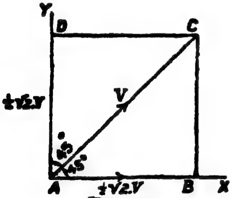

Fig. 27.

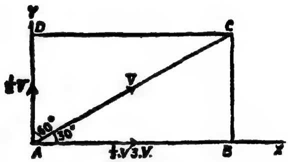

Fig. 28.

CASE II.—*A velocity V is inclined at angles 30° and 60° to two given perpendicular lines. To resolve it into components along these lines.*

Make the same construction as before.

Since $\angle BAC = 30°$ and $\angle ACB = 60°$, therefore ABC is a semi-equilateral triangle; [Appendix, § 6]

$$\therefore \quad \frac{BC}{1} = \frac{AB}{\sqrt{3}} = \frac{AC}{2};$$

$$\therefore \quad AB = \tfrac{1}{2}\sqrt{3}\,.\,AC \text{ and } BC = \tfrac{1}{2}AC.$$

Hence the required components are $\frac{1}{2}\sqrt{3}.V$ along AB and $\frac{1}{2}V$ along AD. We notice that the greater component $\frac{1}{2}\sqrt{3}.V$ is in the direction with which V makes the smaller angle 30°.

[If therefore the given velocity were to make angles of 60° with AB and 30° with AD, the components would be $\frac{1}{2}V$ along AB and $\frac{1}{2}\sqrt{3}.V$ along AD.]

136. We thus have the following results, *which should be remembered* :—

(i) For angles 45° and 45°

components are $\frac{1}{2}\sqrt{2}.V$ and $\frac{1}{2}\sqrt{2}.V$;

(ii) For angles 30° and 60°

components are $\frac{1}{2}\sqrt{3}.V$ and $\frac{1}{2}.V$.

[NOTE.—If the given velocity V makes angles of A and $90° - A$ with two given perpendicular lines, it may be shown by Trigonometry that the components of V along these lines are $V \cos A$ and $V \sin A$ respectively.]

We may apply these results to find the resultant of two velocities whose directions are inclined at any of the common angles 30°, 45°, 60°, etc., as in the following—

Example.—To find the resultant of two velocities of 5 feet per second and 4 feet per second, whose directions include an angle of 60°.

Resolve the velocity 4 into its components along and perpendicular to the velocity 5. These are $\frac{1}{2}.4$ and $\frac{1}{2}\sqrt{3}.4$, *i.e.* 2 and $2\sqrt{3}$ foot-second units respectively.

Hence the two velocities are together equivalent to $2 + 5$ and $2\sqrt{3}$ units in these directions, which are at right angles. Therefore, by § 133, their resultant

$$= \sqrt{\{7^2 + (2\sqrt{3})^2\}} = \sqrt{(49 + 12)} = \sqrt{61} = 7{\cdot}81 \text{ feet per second approximately.}$$

EXAMPLES XI.

1. One body moves due north at the uniform rate of 1·4 feet per second, and another moves due east from the same point at the uniform rate of 4·8 feet per second. Both started at the same instant. Find their relative velocity, and their distance apart half a minute after starting.

2. Two men start at the same time to walk along two streets running at right angles to each other. One of the men walks at the rate of 4 miles an hour and the other at the rate of 3 miles an hour. Find their distance apart at the end of $1\frac{1}{2}$ minutes.

3. A train is receding from an observer with a velocity due south, and the observer himself is seated in another train which is moving due east at the same speed as the first. In what direction does the first train appear to move?

4. A carriage is travelling through a shower of rain which appears to meet the carriage at an angle of 30° to the vertical with a velocity equal to twice that of the carriage. Find the actual direction and velocity of the rain.

5. A boat is rowed on a river so that its speed in still water would be 6 miles an hour. If the river flows at the rate of 4 miles an hour, show, by drawing a figure, how to find the direction in which the head of the boat must be kept in order that its motion may be at right angles to the current.

6. A ship is sailing north-east with a velocity of 10 miles an hour, and to a passenger on board the wind appears to blow from the north with a velocity of $10\sqrt{2}$ miles an hour. Find the true velocity of the wind.

7. A person on an express train moving 60 miles an hour wishes to hit a stationary object, which is situated 100 yards off, in a line through the marksman at right angles to the line of motion of the train. If his bullet moves 1,200 feet per second, find how much to one side of the object he should aim.

8. A body is moving due east with a velocity of 24 miles an hour, and another velocity is communicated to it, so that it now moves due south with the same velocity as before. Find the magnitude and direction of the velocity communicated to the body.

9. A carriage is travelling at the rate of 20 feet per second, and a passenger inside projects a ball with a velocity of 40 feet per second in a direction making an angle of 120° with the direction of motion of the carriage. Show that the resultant velocity of the ball is perpendicular to the direction of motion of the carriage, and determine its magnitude.

10. With what velocity must a man swim at right angles to the current across a stream 120 yards wide, flowing 3 miles an hour, so that he may not be carried further down the river than 176 feet?

11. A ship is sailing due north at the rate of 4 feet per second; a current is carrying it due east at the rate of 3 feet per second; and a sailor is climbing a vertical mast at the rate of 2 feet per second. What is the velocity of the ship, and what the velocity of the sailor, relative to the sea bottom?

12. A ship is sailing north at the rate of 8 miles an hour through the sea, and a man walks at the rate of 7 feet per second straight across her level deck on a line drawn at right angles to her length. Draw a diagram (as well as you can to scale) by measuring which one might find the angle the man's resultant path makes with the north, and calculate his velocity with respect to the sea.

13. To the same particle are imparted two velocities of 5 and 12 feet per second respectively in directions at right angles to each other. Find the resultant velocity.

14. Resolve the following velocities into components along and perpendicular to the straight lines with which their directions make the given angles :—

(i) 4 feet per second, 0°; (v) 24 feet per second, 90°;
(ii) 15 miles an hour, 30°; (vi) 10 kilometres per hour, 120°;
(iii) 12 yards per minute, 45°; (vii) $4\sqrt{2}$ miles an hour, 135°;
(iv) 10 metres per minute, 60°; (viii) 12 feet per second, 150°;
(ix) 10 centimetres per second, 180°.

15. A body has a velocity of 7 miles an hour to the north, and also a velocity of $3\sqrt{2}$ miles an hour to the south-east. It is brought to rest by a third velocity. Determine the magnitude of this velocity.

16. Show how to find, by a graphical construction drawn as well as you can to scale, or by calculation, the resultant of the following velocities, which are communicated to a point, viz. 10 feet per second in an easterly, 20 feet per second in a north-easterly, and 30 feet per second in a northerly direction, respectively.

17. Find the resultant velocity of a body which has communicated to it simultaneously the following velocities: 35 feet per second to the north, 30 feet per second in a direction 30° north of east, $30\sqrt{3}$ feet per second in a direction 30° west of south, and 12 feet per second to the west.

CHAPTER XII.

THE PARALLELOGRAM OF ACCELERATIONS.

137. **General Definition of Acceleration.**—When a body is moving in a straight line, its acceleration, if it has any, must be in the line of motion, and may be defined as in Chap. II. In all other cases we must define the measure of acceleration as follows :—

DEFINITION.—**Acceleration** is measured by the *rate per unit time at which velocity is being acquired,* and the direction of the acceleration is the *direction of this acquired velocity.* The *velocity acquired* by a body in any interval of time is that velocity which must be compounded with the initial velocity in order to obtain the final velocity, *the composition being effected by the Parallelogram (or Triangle) of Velocities.*

From this definition it will be seen that changes in the direction of motion of a body involve acceleration, as well as changes in its actual speed. When a body is moving in a curve, the direction of its acceleration will be different from the direction of motion at any instant.

Examples.—(1) A body is thrown with a velocity of 96 f.s. at an inclination of 30° to the horizon, and moves under gravity. When will it cease to rise, and what velocity will it have?

The velocity after t seconds is compounded of the original velocity and gt vertically downwards. Replacing the former by its horizontal and vertical components, as in § 135, Case II., we see that the components at time t are $96 \times \frac{1}{2}\sqrt{3}$ and $96 \times \frac{1}{2} - gt$.

Hence the body will be moving horizontally when $48 - gt = 0$ or $t = 1\frac{1}{2}$ sec., and its velocity will be $48\sqrt{3}$ or 83·14 . . . f.s.

(2) At a certain instant, a particle is moving due north with a velocity 3, and a second later it is moving north-east with a velocity $3\sqrt{2}$. Find its acceleration.

Let **OA** and **OB** represent the first and last velocities; then **AB** will represent the *velocity acquired* in the second, or the acceleration.

It is easy to prove by Trigonometry, or, indirectly, by Geometry, that the triangle **OAB** is a right-angled isosceles triangle (Appendix, § 5), and that **AB** is equal to **AO**. Hence the acceleration is 3, towards the east.

Note that the particle does not *travel along* any of the lines of the figure. If it starts from **O**, it begins to move along **OA**, but is constantly deflected until finally it is moving parallel to **OB**. That is all we know about it at present.

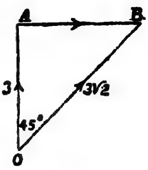

Fig. 29.

(3) A body, uniformly accelerated, starts with a speed of 20 feet per second in a direction 30° west of south, and 10 seconds later it is moving with the same speed in a direction 30° east of south. To find the acceleration and the velocity of the body 5 seconds after starting.

Draw **AD** due south. Make $\angle$**BAD** = $\angle$**DAC** = 30°, and take **AB** = **AC** = 20 units of length (Fig. 30). Then **AB**, **AC** represent the initial and final velocities of the body, and, by the Triangle of Velocities, **BC** represents the velocity which must be compounded with the former to obtain the latter. **BC** therefore represents the *change of velocity* in 10 seconds.

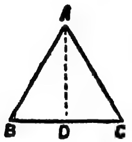

Fig. 30.

Since **AB** = **AC** and $\angle$**BAC** = 60°, the triangle **ABC** is equilateral, and **AD**, the bisector of **BAC**, bisects the base **BC** at right angles. Thus **BC** = **AB** = 20, and **BC** points due east.

Therefore the velocity acquired in 10 seconds is 20 feet per second in a direction due east, and therefore the body is subject to an eastward acceleration of 2 feet per second per second.

The actual change of position of the body in 10 seconds is represented by a length of 10 times **AD** measured along **AD**, as may be verified after § 144 has been read.

At 5 seconds from starting, the acquired velocity is half as great, and is represented by **BD**. Therefore the actual velocity is represented by **AD**. Since $\angle$**BAD** = 30°, therefore (Appendix, § 6)

$$\mathbf{AD} = \mathbf{AB}\,\frac{\sqrt{3}}{2} = 20\,\frac{\sqrt{3}}{2} = 10\sqrt{3}.$$

Hence the velocity 5 seconds after starting is $10\sqrt{3}$ feet per second due south.

138. When two bodies have the same acceleration, their relative velocity is uniform.

Let **AB**, **AC** represent the initial velocities of two bodies at any instant. Then **BC** represents their initial relative velocity (§ 126).

Let **OA**, drawn *towards* **A**, represent the velocity acquired by either body in *any* given interval of time, under the common acceleration.

Then the final velocities are obtained by compounding the velocity **OA** with **AB** and **AC**, respectively; and are, therefore, represented by **OB**, **OC**. Hence the final relative velocity is represented by **BC**, and is the same as the initial relative velocity. Therefore the relative velocity is constant.

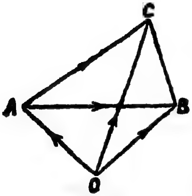

Fig. 31.

Another way of stating this principle is as follows: *If two bodies are moving in any way, their relative motion will not be altered by a common acceleration being imparted to them* (compare §§ 59, 127).

Cor. *If two bodies are projected in any directions and fall under gravity, their relative velocity will be the same as if gravity were not acting.*

This property is of frequent use in investigating the motion of projectiles.

139. Properties of Velocities Extended to Accelerations.—From the fact that an acceleration is a velocity acquired per unit time, it follows that, to most of the properties of velocities proved in the last chapter there correspond analogous properties of accelerations. These we shall now enumerate, in some cases without proof.

An acceleration may be represented by a straight line, for the velocity imparted *per* unit of time may be represented by a straight line, and we may take this line to represent the acceleration.

Thus an acceleration of f feet per second per second in any direction may be represented by drawing a line in that direction, and on it measuring a length representing f feet.

The acceleration of one body relative to another is the rate of change of their relative velocity *per* unit time, defined as in § 137.

It is also the acceleration with which the first body *would appear* to move, if observed by a person moving with the second body.

Let AB, AC represent the velocities acquired by the two bodies per unit time. Then, by § 127, BC represents the relative velocity acquired per unit time, *i.e.* the relative acceleration.

The component and resultant accelerations of a body are the rates of change of the component and resultant velocities *per* unit of time, defined as in § 137.

140. The Parallelogram of Accelerations.—*If two component accelerations be represented by two adjacent sides of a parallelogram drawn from a point, their resultant acceleration shall be represented by the diagonal of the parallelogram drawn from the same point.*

For since the sides of the parallelogram represent the component accelerations, they represent the component velocities acquired by the moving body per unit time. By the Parallelogram of Velocities, therefore, the diagonal represents the resultant velocity acquired per unit time, due to the two components, and this is the resultant acceleration of the body.

The following are easy deductions from the Parallelogram of Accelerations or the Triangle and Polygon of Velocities. Notice the difference between these enunciations and those of §§ 131, 132.

141. Triangle of Accelerations.—*If three accelerations be represented by the sides of a triangle taken in order, then a body whose acceleration is compounded of the three will* ***either remain at rest or move uniformly in a straight line.***

Polygon of Accelerations.—*Generally, if a body have any number of component accelerations, represented by the sides of a closed polygon taken in order, the body* ***either remains at rest or moves uniformly in a straight line.***

For, in either case, the sides representing the accelerations also represent the component velocities imparted per unit of

time. By the Triangle or Polygon of Velocities the resultant imparted velocity is zero.

142. Composition of Two Accelerations at Right Angles.—If f_1, f_2 be the component accelerations in two directions at right angles, F the resultant acceleration, then, as in § 133,

$$F^2 = f_1^2 + f_2^2.$$

[*Resolution of a given acceleration in two directions at right angles.*—Similar results to those proved in § 135 hold for accelerations, but they are not so often used.]

143. Projectiles.—The properties of accelerations enable us to investigate the motion of a body projected in any direction and falling under gravity. Such a body may be called a *projectile.* We shall always neglect the resistance of the air, and shall assume that the acceleration of gravity (g) is the same (both in magnitude and direction) at all points of the path.

Problems of this class are solved by using the "**Principle of Independence of Motions**" (or the "Principle of the Physical Independence of Forces"), which was given by Newton as a sort of rider to his Second Law of Motion, and is supposed to be implied in the expression "*change* of motion." We may state it thus: "If a force acts upon a body in motion, the change of motion produced by the force is the same as if the body were at rest." In other words, to find the final position of a body that is under two (or more) *influences*, let each influence be supposed to have its full effect, one acting after the other.

The influences may be two velocities, as in the Parallelogram of Velocities, or a velocity and an acceleration (as in the case we are coming to), or two accelerations, as in the Parallelogram of Accelerations; or, finally, any more elaborate combination of velocities and accelerations. Of course, an *acceleration* is always caused by a *force*.

144. A body is thrown with a given velocity V in any given direction. To construct geometrically its position at any given instant of the motion.

Let AP be the direction of projection.

On **AP** cut off

AB $= Vt =$ distance which would be traversed in time t, if the velocity were uniform and equal to V.

Draw **AD** vertically downwards, and make

AD $= \frac{1}{2}gt^2 =$ distance which would be traversed in time t by a body falling from rest at **A**.

Complete the parallelogram **ABCD**. Then **C** represents the actual position of the body at the time t.

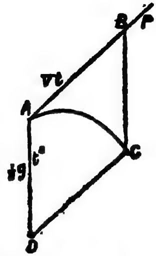

Fig. 32.

This follows immediately from the last article. For **AB** is the space due to the velocity of projection, and **AD**, or **BC**, is the space due to gravity. (It is obvious that the two lines **AB** and **BC** alone are sufficient to fix the position of the body.)

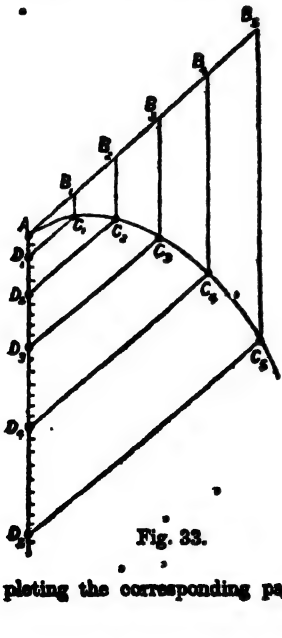

Fig. 33.

The *path* of the body is not, in this case, along the straight line **AC**, but along a curve touching **AB** at **A**, as will be shown more clearly below.

The *velocity* of the body after 1, 2, 3, ... t seconds can be determined by proceeding as in § 130, but *a separate diagram will have to be drawn*. If we measure off, successively, lengths of V units and gt units parallel to **AB**, **BC**, respectively, the third side of the triangle thus found represents the velocity at time t. (§ 145.)

Fig. 33 shows how the construction of Fig. 32 is used to find the position of the body at every second of the motion. The points D_1, D_2, D_3 ... represent the positions of a body falling from rest after 1, 2, 3 . . . seconds respectively. They are therefore the points shown in the diagram on page 37. On the direction of projection, we must take each of the divisions AB_1, B_1B_2, B_2B_3 ... to represent V feet (supposing the velocity V to be measured in feet per second), or the distance traversed in one second with velocity V. Completing the corresponding parallelograms, we find the points C_1, C_2,

C_3 ... representing the positions of the projectile after 1, 2, 3 ... seconds respectively.

If the points A, C_1, C_2, C_3 ... be joined together by a curve, this curve, when well drawn, will represent the path described by the projectile. The curve must be drawn *touching* AB at A, for, at the instant of projection, the direction of motion is *along* AB.

[The curve is called a *parabola*.]

Example.—A stone is projected horizontally from the top of a tower 80 feet high, with a velocity 30 f.s. Find where it will strike the ground, and with what velocity.

Considering the horizontal and vertical motions independently,

time occupied in fall (by $s = \frac{1}{2}gt^2$) $= \sqrt{5}$ seconds.

Horizontal space described in this time with a velocity 30 f.s.

$= 30\sqrt{5}$ feet.

The stone therefore strikes the ground at $30\sqrt{5}$ feet from the foot of the tower.

The vertical velocity $= 32\sqrt{5}$ (by $v = gt$ or $v^2 = 2gh$), and the horizontal velocity $= 30$;

$\therefore$ resultant velocity $= \sqrt{\{30^2 + (32\sqrt{5})^2\}} = \sqrt{(6020)} = 77{\cdot}6$ f.s.

145. To find geometrically the velocity of a projectile after time t seconds.

To find this we must compound the two velocities—viz. the original velocity (V) represented by AB (Fig. 33*a*), and the downward velocity (gt) produced by the earth's attraction in t seconds (represented by AC).

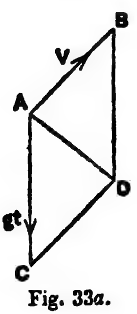

Fig. 33*a*.

Using the Parallelogram Law we find the resultant AD.

NOTE 1.—AD represents the required velocity in magnitude and direction, but the projectile is not travelling along the line AD.

NOTE 2.—Contrast Figs. 32, 33*a*. The former gives the *position* after t seconds; the latter gives the *velocity* after t seconds.

145*a*. Vertical and Horizontal Velocities.—Another method of dealing with projectiles (*i.e.* finding position and velocity) is to resolve the given initial velocity into its horizontal and vertical components, say X and Y respectively.

Now suppose that after a time t seconds the particle is at **P**. Then we call **AN** the horizontal displacement and **NP** the vertical displacement in t seconds.

Now the vertical motion is due to the initial vertical velocity Y and the acceleration of gravity (g ft./sec.^{2} downwards).

Thus—

vertical velocity after time $t = Y - gt$ [§ 52 (7)

and **NP** = the *vertical displacement in time* $t = Yt - \frac{1}{2}gt^2$ [§ 52 (8)

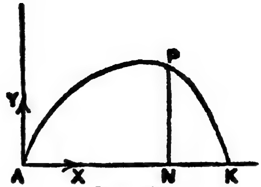

Fig. 33b.

Now the acceleration, being entirely vertical, has no effect on the horizontal velocity X, which therefore remains uniform

Hence— *horizontal velocity after time* $t = X$,

and **AN** = *horizontal displacement in time* $t = Xt$ [§ 14 (1)

AN and **NP** determine the actual *position* **P**. To find the actual *velocity* at this moment we must combine the two component velocities determined above, viz. X (horizontal) and $Y - gt$ (vertical).

NOTE.—If another particle is projected from **A**, with vertical velocity Y, at the same instant as the given particle is projected with velocity components Y and X, then the two particles will always be at the same height.

Example.—A cricket ball is thrown up with a velocity whose horizontal and vertical components are 30 f.s. and 40 f.s. respectively. To what height does it rise, and after what time does it fall back to the ground?

Suppose a second cricket ball thrown up from the same point at the same time with vertical velocity 40 ft. per sec. Then the two balls will always be at the same height.

Thus we need only find to what height the second ball rises and when it falls back to the ground.

By § 54 the required height $= \frac{u^2}{2g} = \frac{40^2}{64} = 25$ ft.

By § 55 the required time $= \frac{2u}{g} = \frac{2 \times 40}{32} = 2\frac{1}{2}$ sec.

EXAMPLES XII.

1. If a body be moving north with a velocity of $5\sqrt{2}$ feet per second, and after 5 seconds it is found to be moving at the same rate eastward, what are the direction and magnitude of the acceleration, supposed uniform?

2. A body is moving at the rate of 12 feet per second, and after 6 seconds it is found to be moving at the same rate, but in a direction inclined at an angle of 60° to its former one. Supposing the acceleration be uniform, how will the body be moving after 6 seconds more?

3. A particle of mass m moves uniformly with velocity v along the sides of a square. Calculate the change of velocity at each corner of the square, and the magnitude of the blow, measured by the total change of momentum, required to cause this change.

4. A body is projected horizontally from the top of a tower with a velocity of 32 feet per second. Represent in a diagram its velocities after 1, 2, 3 seconds respectively, and find their magnitudes.

5. A particle is projected in a horizontal direction with a velocity of 10 miles an hour, and at the same time falls under gravity. Assuming that no other forces are acting, and taking $g = 32$ (feet, seconds), draw a picture representing the position of the particle at the end of 1, $1\frac{1}{2}$, $2\frac{1}{2}$, and 3 seconds.

6. A cannon ball is fired horizontally from the top of a tower 49 feet high, with a velocity of 200 feet per second. Find at what distance from the tower the cannon ball will strike the ground.

7. A person inside clings to the roof of a railway carriage, which then rushes horizontally over the edge of a precipice. What change, if any, in his motion will result if he lets go his hold? Give a reason for your reply.

8. A body is projected horizontally from the top of a tower with a velocity of 300 feet per second. It strikes the ground in 5 seconds. What is its distance from the foot of the tower?

9. A balloon is carried along at a height of 100 feet from the ground, with a velocity of 40 miles an hour, and a stone is dropped from it. Find the time before the stone reaches the ground, and the distance from the point where it reaches the ground to the point vertically below the point where it left the balloon.

10. A ball is thrown horizontally from a height of 225 feet, with a velocity of 90 feet per second. What is its velocity on reaching the ground?

11. A balloon is carried along by a current of air, moving from east to west, at the rate of 60 miles an hour, and a stone is dropped from it. What sort of a path will it appear to describe, as seen by a man in the balloon?

12. A body is projected horizontally from the top of a tower, with a velocity of 60 feet per second, and in 5 seconds it strikes the ground. Find the distance between the point of fall and the point of projection.

13. A cannon ball of mass 14 lb. is fired horizontally from a gun whose mass is 4 tons. The mouth of the cannon is 16 feet from the ground, and the ball strikes the ground 880 feet off. What force will be required to bring the cannon to rest in 10 feet?

14. A body is projected upwards with a velocity whose vertical and horizontal components are 38 and 8 feet per second. Find its distance from the point of projection after 2 seconds.

15. A stone is projected horizontally with a velocity of $32\sqrt{3}$ feet per second. Find what time must elapse before it is moving with a velocity of 64 feet per second.

16. A cannon ball is projected into the air in a direction inclined at an angle of 30° to the horizon. If the initial velocity of the ball be 1000 feet per second, at what distance from the cannon will it strike the ground?

17. A body is projected upwards in a direction inclined at 60° to the horizon. Show that its velocity when at its greatest height is half of its initial velocity.

18. A stone is projected into the air with a velocity of 200 feet per second in a direction inclined at 60° to a horizontal plane. With what velocity must another stone be projected vertically upwards so that the two stones may rise to the same height above the horizontal plane?

EXAMINATION PAPER VII.

1. Explain and illustrate the proposition known as the "Parallelogram of Velocities."

2. If a point has a velocity of 1 foot per second to the east, and also a velocity of $\sqrt{3}$ per second to the north, determine the velocity which must be compounded with these to bring the point to rest.

3. A body is moving from north to south with a velocity of 30 miles an hour, while at the same time it moves from east to west with a velocity of 33 feet per second. Find the direction of motion, and the velocity of the body in yards per minute.

4. A river, ½ mile wide, runs at the rate of 3 miles an hour. A boat, rowed at 3 miles an hour relative to the water, is required to reach the opposite bank at a point ½ mile lower down. Find the direction in which the boat must be steered, and the time occupied in crossing.

5. A man walks across the deck of a ship, sailing due north at the rate of 10 miles an hour, in 9 seconds, and finds he has actually moved in a direction 30° east of north. How wide is the deck?

6. Show that, when two bodies are equally accelerated, their relative velocity is uniform.

7. A straight stick PQ moves on a table parallel to itself in such a way that the end P traces out, with uniform velocity of 5 feet per minute, a straight line inclined at an angle of 60° to PQ. At the same time a fly walks along the stick at the uniform rate of 5 feet per minute. Find how fast the fly walks, relative to the table.

8. Show how to find, at any given instant, the position and velocity of a body which is projected with a velocity whose horizontal and vertical components are u and v.

9. A body is projected with an upward vertical velocity of 16·6 feet per second, and a horizontal velocity of ·8 foot per second. Show that its distance from the point of projection at the end of 1 second is 1 foot.

10. A balloon is moving horizontally with a velocity of 30 miles an hour, and a stone is projected horizontally from it with a velocity of 14 feet per second in the direction opposite to that in which the balloon is moving. The stone reaches the ground in 4 seconds. Find the height of the balloon, and the distance of the point where the stone reaches the ground from the point vertically under the balloon at the instant of projection.

CHAPTER XIII.

THE PARALLELOGRAM OF FORCES.

146. Representation of Forces by Straight Lines.—Newton's Second Law tells us that force, like velocity, has direction as well as magnitude. For it shows us that rate of change of momentum is proportional to the force, *and takes place in the direction in which the force is impressed.* Hence the magnitude of a force is measured, as in Chap. VI., by the momentum per unit time which it imparts to the body on which it acts, and the direction of the force is the direction of this imparted momentum.

Or, what is equivalent, the magnitude of the force may be measured by the *velocity* it would impart to a *unit mass* in unit time, and its direction is the direction of this velocity, or the direction in which the body would begin to move if it started from rest.

If, therefore, this velocity be represented by a straight line, this line will indicate both the magnitude and direction of the force, and it may therefore be said to represent the force.

Thus forces may be represented by straight lines. (See also § 148.)

147. The Principle of the Physical Independence of Forces.—When a body, instead of starting from rest, is initially moving in a direction different from that of the impressed force, the velocity which the force imparts to the body must be compounded with the body's initial velocity in order to obtain its final velocity.

A few simple illustrations will show this.

(1) Let A, D be the positions at any instant of two men seated in a railway carriage moving uniformly with velocity AB. If the man at A throws a ball so as to reach the other in one second, *he will project it*

in the direction **AD**, *in just the same way as he would have done if the carriage had been at rest.* But, owing to the motion of the train, the two men will, in one second, be carried, say, to **B**, **C**, and the actual path of the ball in space will be the diagonal **AC**.

Hence the force exerted in throwing the ball merely imparts the *relative* velocity **AD**. But, before the ball was thrown, it had the same velocity **AB** as the carriage. Therefore the final velocity **AC** is obtained by compounding the initial velocity **AB** with the velocity **AD** due to the impressed force of projection.

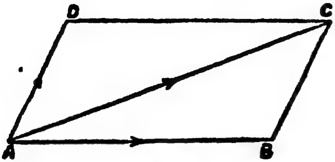

Fig. 34.

(2) Again, when a stone is dropped, we say that its momentum is equal to the product of the time into its weight. But, when the Earth's motion is taken into account, the momentum which we observe is only the momentum of the velocity *relative* to the Earth. The actual velocity of the stone in space is compounded of the velocity of the Earth and this relative velocity. That is, its final velocity is compounded of its initial velocity and the velocity due to its weight, the latter component being given by Newton's Second Law.

This property may be stated more generally thus: *The velocity-component which any given force imparts to a body in any given time is independent of any other velocity-components which the body may possess or acquire.*

This is called the **Principle of the Physical Independence of Forces.**

Employing the definition of acceleration of § 137, it hence follows that the relation

$$P = mf,$$

or

$$\text{force} = \text{mass} \times \text{acceleration},$$

holds good in every case of motion under force.

148. **Point of Application of a Force.**—*A force cannot act on nothing;* it must be applied to some definite particle or body whose velocity it changes or tends to change, and the change of velocity will depend on the mass moved. Hence, to completely define a force, it is necessary to specify on what particle the force acts; *i.e.* to specify its *point of application* (§ 105).

When, therefore, a force is represented by a straight line, this line must be drawn from its point of application.

149. Composition of Forces Acting on a Particle.—A body may be acted on by two or more independent forces at the same time; in fact, it generally is so.

If we lift a body off the ground, the body is acted on simultaneously by two *entirely distinct* forces, namely, its weight and the lifting force exerted by our hand.

When two or more forces act simultaneously *on the same particle*, each force tends to impart a certain acceleration in the direction in which it is applied. But a particle cannot actually move in two different ways at the same time; it must move with a certain definite acceleration in some direction. Such an acceleration could always be produced by a single force of suitable magnitude applied to the particle in that direction. This force is called the *resultant* of the original system of forces. If the acceleration vanishes, the forces are said to be in *equilibrium*. Hence we have the following

DEFINITION.—The **resultant** of two or more forces is *that force which would produce the same acceleration that is produced by the several forces acting simultaneously.*

Any forces which have a given force for their resultant are called **components** of the given force.

DEFINITION.—A system of two or more forces is said to *balance*, or to be in **equilibrium**, *when the forces, acting simultaneously, produce no change in the state of rest or uniform motion of the body or bodies to which they are applied.*

150. The Parallelogram of Forces.—**If two forces, acting simultaneously on the same particle, be represented by two adjacent sides of a parallelogram drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point.**

Let the two forces, P, Q be represented by the sides **AB**, **AD** of the parallelogram **ABCD**. These lines represent (§ 146)

the velocities which P and Q, acting separately, would impart to a unit mass in a unit time. When the two forces act on the same particle during the same time, the velocity-component imparted by either force is independent of that imparted by the other (§ 147). Therefore the actual velocity acquired is found by compounding the velocities **AB**, **AD** by the Parallelogram of Velocities, and is therefore represented by the diagonal **AC**. Hence the change of momentum is the same as would be produced in the same time by a single force represented by **AC**; therefore the diagonal **AC** represents the resultant of the two given forces, as was to be proved.

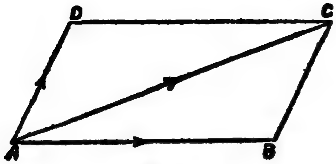

Fig. 34.

151. Deductions from the Parallelogram of Forces.—The following properties of forces acting on a particle are analogous to those of velocities and accelerations (§§ 131-136 and 141, 142). As they will be considered more fully in treating of Statics, we shall now merely state them without proof.

Triangle of Forces.—If three forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle taken in order, they will be in equilibrium.

Polygon of Forces.—If any number of forces acting on the same particle can be represented in magnitude and direction by the sides of a closed polygon taken in order, they will be in equilibrium.

Composition of Two Forces at Right Angles.—If X and Y denote two forces acting at right angles on a particle, the magnitude of their resultant R is given by $R^2 = X^2 + Y^2$.

Resolution of a Force in Two Directions at Right Angles.—The results of §§ 135, 136 apply to forces.

EXAMPLES XIII.

[Further examples on Composition and Resolution of Forces will be given in Statics. The following are miscellaneous examples.]

1. Find the resultants of the following pairs of forces acting at right angles to one another:—

(i) 10 lb. and 24 lb.; (iii) 8 grammes and 15 grammes;
(ii) 20 oz. and 21 oz.; (iv) 21 tons and 220 tons.

2. Three forces of 5, 12, and 13 lb. respectively, act on a particle; what are the greatest and least values of their resultant?

3. Two bodies start together from rest, and move in directions at right angles to each other. One moves uniformly with a velocity of 3 feet per second; the other moves under the action of a constant force. Determine the acceleration due to this force, if the bodies at the end of 4 seconds are 20 feet apart.

4. The horizontal and vertical components of a certain force are equal to the weights of 5 and 12 lb. respectively. What is the magnitude of the force?

5. Supposing this force (see Question 4) to act for 10 seconds on a mass of 8 lb., which is also exposed to the action of gravity and is initially at rest, what velocity will be communicated to the mass, the vertical component of the force acting upwards?

6. A bullet is dropped from a height of 9 feet above the ground by a man in a train moving at the rate of 60 miles an hour. Find how far the train will have moved before the bullet reaches the ground.

CHAPTER XIV.

MOTION DOWN INCLINED PLANES.

152. Definitions.—An inclined plane may be exemplified by a plank tilted up at one end, so that bodies can slide down it, or by a road or railway running down hill at a uniform slope. It will, however, be convenient to take an inclined plane as the slanting face ACC′A′ of a block of material whose vertical face ABC is a right-angled triangle. The hypotenuse AC is called the **length** of the plane, AB is the **base** and is horizontal, the perpendicular BC is the **height** of the plane, and the angle BAC measures its **inclination** to the horizon.

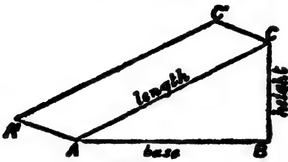

Fig. 35.

The plane is said to be at an inclination of "p **in** q," when its height is to its length as p to q, or

$$\frac{BC}{p} = \frac{AC}{q};$$

so that there is a rise or fall of p feet for every q feet traversed up or down the plane.

Thus, if $\angle BAC = 30°$, the inclination is 1 in 2 (Appendix, § 6).

By a **smooth** plane or other surface we mean one that is perfectly slippery or devoid of friction, so that bodies can slide along it without resistance. The surface exerts a reaction, for otherwise the body would penetrate it; *but the reaction of a smooth surface is wholly perpendicular to the surface* (§ 78).

153. To find the acceleration of a body sliding down a smooth incline of 1 in n.

Let the body be on the plane at **C**. Let its weight be represented by the vertical line **Ca**. Draw ab perpendicular to the plane.

Then **Cb**, ba represent the components of the weight along and perpendicular to the plane.

Since the body moves down the plane, the *resultant force* producing motion is down the plane. Hence the reaction of the plane, acting perpendicular to it, must be represented by ab, and the force producing motion by **Cb**

On the plane cut off

$$\text{CA} = \text{C}a,$$

and draw **AB** horizontal. Then the right-angled triangles **ABC**, ab**C** are equal in every respect;

$$\therefore \quad \text{C}b = \text{CB}.$$

But, since the incline is 1 in n, the height **CB** is one nth of the length **CA**; therefore also

$$\text{C}b = \text{C}a \div n;$$

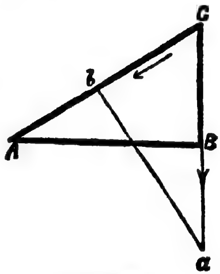

Fig. 36.

$\therefore$ resultant force producing motion = weight of body $\div n$;

$$\therefore \quad \text{the acceleration down the plane} = \frac{g}{n} \quad \ldots\ldots\ldots \quad (1)$$

$$= g \times \frac{\text{height of plane}}{\text{length of plane}}.$$

In particular, if the inclination = 0°, 30°, 45°, 60°, 90°,

the acceleration down the plane $= 0, \frac{g}{2}, \frac{g\sqrt{2}}{2}, \frac{g\sqrt{3}}{2}, g,$

(by § 135, or Appendix, §§ 5, 6).

For experimental verification see § 321, *Exp.* 1 (*Note*).

154. A heavy body slides from rest down a smooth inclined plane. To construct its position at a given time t.

Let the body start from rest at **A**. Draw **AC** vertically downwards, and cut off

AC $= \frac{1}{2}gt^2 =$ distance that would be fallen in time t by a body dropped from **A**.

Drop **CB** perpendicular on the plane **AB**.

Then **B** will represent the position of the body on the plane at the time t.

For, if it were acted on by gravity alone, it would be at **C**.

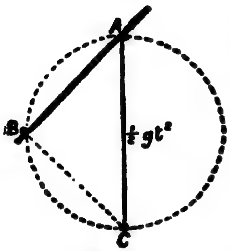

Fig. 37.

The only other force, namely, the reaction of the plane, is always perpendicular to the plane, and therefore parallel to **CB**; hence it can only produce motion of the particle in the direction **CB**.

Therefore the particle is in **CB**.

But it is also in the plane **AB**.

Therefore it is at **B**.

NOTE.—If the inclination is 1 in n, it is easy to see that **AB** is $1/n$th of **AC**, a result which at once follows from the fact that the acceleration is g/n.

*COR. 1. Since the angle **ABC** is a right angle, **B** lies on a circle having **AC** as diameter (Euc. III. 31). Hence, *if any number of bodies start simultaneously from* **A**, *and slide down straight lines in the same vertical plane, their positions at any instant will all lie on a circle whose highest point is* **A**.

*COR. 2. Hence *the times taken to slide down different chords of a vertical circle, starting from the highest point of the circle, are all equal.*

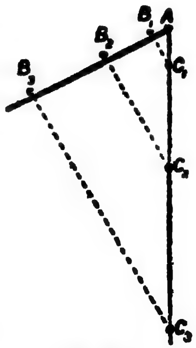

Fig. 38.

155. Fig. 38 shows how the position of the body may be constructed at each second of the motion. The points C_1, C_2, C_3 are the positions of a freely falling body after 1, 2, 3 seconds, *and these are*

given by the diagram on page 38. Drawing perpendiculars on the plane, their feet B_1, B_2, B_3 represent the positions of a body sliding down the plane, at the same instants.

156. Work.—When the direction of motion of a body is not in the same straight line with the force acting on it the work done must be defined as follows :—

DEFINITION.—Let a force P, constant in magnitude and direction, move its point of application from **A** to **C**. Draw **CB** perpendicular on the direction of P. Then the product of the force P into the distance **AB** measures the **work done** by the force, **AB** being considered positive or negative according as its direction is the same or opposite to that of the force. *When the point of application moves perpendicular to the force, no work is done.*

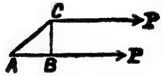

Fig. 39.

Thus, if the point of application were moved from **C** to **A**, the work done would be $P \times$ **BA**, or $P \times (-$**AB**$)$.

If the point of application is moved first from **A** to **B** and then from **B** to **C**, the work done by P in the former displacement is $P \times$ **AB**, and in the latter it is zero, because **BC** is perpendicular to P; therefore the whole work done is $P \times$ **AB**, the same as if the point of application moved directly from **A** to **C**.

157. Work on an Inclined Plane: Work done by Gravity.—When a weight W slides down the inclined plane **CA** (Fig. 35 or 36), the work done by gravity is by definition,

$$= W \times \mathbf{CB} = W \times \text{vertical height descended};$$

and is the same as the work which would be done in falling vertically down the height of the plane.

Thus *the work done by gravity on a body is always equal to the product of the weight of the body into the vertical height through which it descends*, whether the weight falls vertically or slides down an inclined plane.

Similarly, the work done *against* gravity in raising a body is the product of the weight into the vertical height through which it is raised.

When a body moves horizontally no work is done either by or against gravity.

Examples.—(1) To find the work done against gravity by a horse in pulling a cart weighing 5 cwt. up a hill a mile long, at a slope of 1 in 40.

$$\text{Vertical height} = \frac{1}{40} \text{ of a mile} = \frac{5280}{40} \text{ ft.} = 132 \text{ ft.,}$$

$$\text{weight raised} = 5 \times 112 \text{ lb.} \quad = 560 \text{ lb.};$$

$$\therefore \quad \text{work done} = 132 \times 560 \quad = 73920 \text{ ft.-lb.}$$

(2) To find the horse-power required to draw a train of 150 tons up an incline of 1 in 128 at 30 miles an hour, if the resistance due to friction is 10 lb. per ton.

In one second the train moves 44 feet;

$$\therefore \quad \text{vertical height risen per sec.} = \frac{44}{128} \text{ ft.} = \frac{11}{32} \text{ ft.}$$

Also, $\quad \text{weight of train} = 150 \times 2240 \text{ lb.};$

$\therefore$ work done per sec. against gravity

$$= \frac{11}{32} \times 150 \times 2240 \text{ ft.-lb.} = 115500 \text{ ft.-lb.};$$

Also, total resistance due to friction $= 10 \times 150 \text{ lb.} = 1500 \text{ lb.};$

$\therefore$ work done per sec. against resistance

$$= 1500 \times 44 \text{ ft.-lb.} = 66000 \text{ ft.-lb.};$$

$$\therefore \quad \text{total work done per sec.} = 115500 + 66000 \text{ ft.-lb.} = 181500 \text{ ft.-lb.};$$

$$\therefore \quad \text{required horse-power} = \frac{181500}{550} = 330.$$

158. To verify the Principle of Conservation of Energy for motion down a smooth inclined plane.

Let a mass m slide down an incline of 1 in n, starting with initial velocity u. By (1), the acceleration is $g \div n$. Hence, if v is the velocity after the body has gone a distance s, then, by (8) § 37,

$$v^2 - u^2 = 2\frac{g}{n}s.$$

$$\therefore \quad \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 = mg \times \frac{s}{n}.$$

The left-hand side represents the increase of kinetic energy. Also mg is the weight of the body, and $s \div n$ is the vertical height fallen; hence the right-hand side represents the decrease of potential energy.

These are, therefore, equal; as was to be proved.

COR. If the body starts from rest, we have

$$v^2 = 2g\frac{s}{n} = 2g \times \text{height fallen.}$$

Hence, *if different bodies slide down inclined planes of the same height, they will all acquire the same speed on reaching the bottom.*

Examples.—(1) A body slides down a smooth plane whose height is one-third its length. To find the velocity acquired when it has travelled 12 feet.

Let the mass of the body be m lb. In travelling 12 ft. it falls a vertical depth of $\frac{12}{3}$ ft. or 4 ft.;

$$\therefore \quad \text{work done by gravity} = 4m \text{ ft.-lb.} = 4mg \text{ ft.-poundals}$$
$$= 4m \times 32 \text{ ft.-poundals.}$$

This is equal to the kinetic energy. Hence, if v is the required velocity,

$$\tfrac{1}{2}mv^2 = 4m \times 32;$$
$$\therefore \quad v^2 = 4 \times 2 \times 32 = 4 \times 64;$$
$$\therefore \quad v = 2 \times 8 = 16 \text{ ft. per sec.}$$

(2) A weight of 3 lb. draws a weight of 4 lb. up an incline of 30° by means of a string passing over a pulley at the top of the plane and hanging vertically. To find the acceleration.

Let v be the velocity acquired when both weights have moved over s feet.

The 3-lb. wt. will have fallen vertically through s ft.

$$\therefore \quad \text{work done by 3-lb. wt.} = 3s \text{ ft.-lb.} = 3gs \text{ ft.-poundals.}$$

The 4-lb. weight will have risen vertically through $\tfrac{1}{2}s$ feet;

$$\therefore \quad \text{work done by 4-lb. wt.} = -4 \times \tfrac{1}{2}s = -2s \text{ ft.-lb.} = -2gs \text{ ft.-poundals.}$$

The whole work done is equal to the kinetic energy;

$$\therefore \quad \tfrac{1}{2}(4+3)\,v^2 = 3gs - 2gs = gs;$$
$$\therefore \quad v^2 = \frac{2g}{7}s.$$

Comparing this with $\qquad v^2 = 2fs,$

we have $\qquad$ required acceleration $f = \tfrac{1}{7}g.$

Alternative Method.—By the method of § 113, it will be seen that

$$\text{acceleration} = (\text{moving force}) \div (\text{mass moved})$$
$$= (3g - 4g \times \tfrac{1}{2}) \div (3+4) = \tfrac{1}{7}g.^*$$

---

* When a heavy body is on an inclined plane, the moving force, or the component of its weight down the plane, is reduced to $mg \sin A$, or mg/n, but its mass (m) of course remains unaltered.

(3) To find the acceleration with which a weight of 4 grammes, sliding down a smooth incline of 1 in 4, draws a weight of 5 grammes along a smooth horizontal plane by a string passing over the edge.

Let v be the velocity acquired after moving s centimetres.

The weight of 4 gm. or $4g$ dynes falls through $\frac{1}{4}s$ cm.;

$\therefore$ kinetic energy = work done $= 4g \times \frac{1}{4}s = gs$ ergs, where $g = 981$;

$$\therefore \quad \tfrac{1}{2}(4+5)\,v^2 = gs, \quad \text{or} \quad v^2 = \tfrac{2}{9}gs.$$

Comparing with $\qquad v^2 = 2fs,$

we have $\qquad f = \frac{1}{9}g = 109$ cm. per sec. per sec.

Or thus: With C.G.S. units, we have

component of weight of mass 4 down the plane $= 4g \div 4 = g$;

$\therefore$ acceleration = (moving force) $\div$ (mass moved) $= g \div (4+5) = 109.$

(4) In a cable tramway, a truck of 5 cwt., descending an incline of 1 in $7\frac{1}{2}$, pulls a truck of 3 cwt. up an incline of 1 in 6. To find the acceleration.

The component of the weight of the first truck down the plane

$$= 5 \div 7\tfrac{1}{2} \text{ cwt.} = \tfrac{2}{3} \text{ cwt.},$$

and this tends to pull the 5-cwt. truck down and the 3-cwt. truck up.

The component of the weight of the second truck

$$= 3 \div 6 \text{ cwt.} = \tfrac{1}{2} \text{ cwt.},$$

tending to pull the 3-cwt. truck down, and the 5-cwt. truck up.

Therefore, on the whole, we have a force of $\frac{2}{3} - \frac{1}{2}$ cwt., or $\frac{1}{6}$ cwt., tending to move a total mass of $(5+3)$ cwt., or 8 cwt.;

$$\therefore \quad \text{required acceleration} = \frac{\frac{1}{6}}{8}g = \frac{g}{48} = \tfrac{2}{3} \text{ ft. per sec. per sec.}$$

EXAMPLES XIV.

1. Find the distances traversed in one second, and the velocities acquired in that time, by particles sliding down smooth inclines of (i) 1 in 16, (ii) 5 in 13, (iii) 3 in 64.

2. Find the distances traversed in 10 seconds, and the velocities acquired in that time, by particles sliding down the given smooth inclined planes:—

   (i) an incline of length 10 feet and height 8 feet;
   (ii) an incline of length 25 feet and base 24 feet;
   (iii) an incline of height 8 feet and base 15 feet.

3. Find the velocities acquired in 1 second by particles sliding down smooth planes of inclinations 30°, 45°, 60°.

4. A boy in a toboggan slides down a perfectly smooth hill, whose inclination is 1 in 20. At what rate will he be going (in miles per hour) when he has travelled 100 yards from the start?

5. A heavy body starting from rest slides down a smooth plane inclined at an angle of 30° to the horizon. How many seconds will it occupy in sliding 240 feet down the plane, and what will be its velocity after traversing this distance ?

6. A body is projected up a smooth inclined plane, whose height is one-half of its length, with a velocity of 60 feet per second, and just reaches the top. Find the length of the plane, and the time taken in the ascent.

7. The height of an inclined plane is three-fifths of its length ; a body is projected up the plane from the bottom with a velocity of 50 feet per second, and slides down again. Find the distance attained, and the time before the body returns to the starting point.

8. A heavy body slides down a smooth plane inclined at 30° to the horizon. Through how many feet will it fall in the fourth second of its motion from rest ?

9. A heavy particle slides from rest down a smooth inclined plane 15 feet long and 12 feet high. What velocity will it possess when it reaches the bottom, and how many seconds will be occupied in the descent ? How long would it have taken to fall vertically through a height of 12 feet ?

10. A stone is projected up a smooth inclined plane with a velocity which will just carry it to the top. How far would a stone ascend a plane of the same length, but twice the height, if projected with half the velocity ?

11. A body, weighing 187 lb., is supported on an inclined plane, whose angle is 30°, by a horizontal force. Find the force and the work necessary to move the body 20 feet along the plane.

12. The pull exerted by a rope which draws a truck up an incline of 1 in 8, with an acceleration of 2 feet per second per second, is 1 ton wt. Find the weight of the truck.

13. Find the work done when a mass of 45 lb. is moved 40 feet up an incline of 30°.

14. Find the h.-p. of an engine which pulls a load of 90 tons at the rate of 30 miles an hour up an incline of 1 in 100, the frictional resistance of the road being 12 lb. per ton.

15. Find at what rate an engine of 15 h.-p. could draw a load weighing 25 tons up an incline of 1 in 280, the resistance from friction being 6 lb. per ton.

16. Find the velocity acquired in 1 second, and the distance traversed in that time, by particles sliding down a plane inclined to the horizon at 30°. (Take $g = 981$.)

17. Find the distances which a particle must slide down inclines of (i) 1 in 9, (ii) 1 in 109, (iii) 1 in 981 respectively, in order to acquire a velocity of 10 centimetres per second.

18. Find, in ergs, the work required to move a mass of 1 gramme through a distance of 1 metre up a plane inclined at 60° to the horizon.

19. If a body slides down a smooth inclined plane, and in the fifth second of its motion from rest passes over 2,207·25 centimetres, find its acceleration and the inclination of the plane to the horizon. (Take $g = 981$.)

20. A smooth inclined plane, whose height is one-half of its length, has a small pulley at the top, over which a string passes. To one end of the string is attached a mass of 12 lb., which rests on the plane, while from the other end, which hangs vertically, is suspended a mass of 8 lb., and the masses are left free to move. Find the acceleration, and the distance traversed from rest by either mass in 5 seconds.

21. A weight of 10 lb. is drawn up a smooth plane inclined at 30° to the horizon by means of a weight which descends vertically, the two weights being connected by a string which passes over the top of the plane. If the acceleration be 8 feet per second per second, find the weight which descends vertically.

22. Two masses m and M are connected by a string which is placed over the top of a plane inclined at 30° to the horizon, so that one of the masses rests on the plane while the other hangs vertically. It is found that m, hanging vertically, can draw M up the plane in half the time in which M, hanging vertically, can draw m up the plane. Find the ratio of m to M.

23. A mass of 6 lb. is connected with another of 18 lb. by means of a fine string which passes over the top of a smooth plane inclined at an angle of 30° to the horizon. The mass of 6 lb. is placed on the plane, and the other mass is placed on a horizontal table whose edge is level with the top of the incline. Find the acceleration produced.

24. In the preceding question find the velocity of either mass when it has traversed a distance of 2 feet from rest.

EXAMINATION PAPER VIII.

1. Give a dynamical proof of the Parallelogram of Forces.

2. Find the resultant of two forces of 8 lb. and 15 lb. acting at a point at right angles to each other.

3. If a body, acted upon by several forces, move in a straight line with uniform velocity, what condition must the forces satisfy?

4. Find the acceleration down a smooth inclined plane.

5. A body slides down a smooth plane inclined at an angle of 30° to the horizon. Find the space described in the first second from rest.

6. Find the space described from rest in 5 seconds by a body sliding down a smooth incline of 7 in 25.

7. A mass of 6 oz. slides down a smooth inclined plane, whose height is half its length, and draws another mass from rest over a distance of 3 feet in 5 seconds, along a smooth horizontal table which is level with the top of the plane, the string passing over the top of the plane. Find the mass on the table.

8. Two bodies, whose masses are P and Q, are connected by a fine string passing over a pulley at the top of a smooth plane inclined at 30° to the horizon. P hangs vertically, and Q rests on the inclined plane. If P descend from rest through a given space in twice the time in which it would fall freely from rest through the same space, find the ratio of P to Q.

9. Show that, if a number of bodies start simultaneously from A and slide down straight lines in the same vertical plane, their positions at any instant all lie on a circle whose highest point is A.

10. A body, whose mass is 1 kilogramma, lies on a smooth plane inclined at an angle of 30° to the horizon. Find the work done against gravity in moving it 1 metre up the plane.

STATICS.

PART I.

EQUILIBRIUM OF FORCES AT A POINT.

CHAPTER XV.

COMPOSITION OF FORCES.

159. **Statics** is that branch of Mechanics which deals with forces applied to a body or a number of bodies which remain **at rest.** Such forces are said to **balance**, or be **in equilibrium.**

For the present we shall only consider the properties of forces applied to a single particle. In Chap. XVIII. we shall treat of forces acting on a body of extended size.

It should not be forgotten, however, that forces are in equilibrium when the body on which they act moves uniformly in a straight line just as well as when it is at rest (§ 149).

Thus, an express train, travelling with uniform speed, is under the influence of a number of forces which form a system in equilibrium.

160. **Force** has been defined (§ 62) as that which changes or tends to change a body's state of rest or motion.

Hitherto, we have looked upon forces in their dynamical aspect, as being measured by the momentum they can generate. *Statically*, we have nothing to do with motion, and a force is practically measured by the weight which it can support. Moreover, there are two main classes of forces, which may be described as **active** and **passive**, the former being competent to originate motion (such as the weight of a body, the push or pull of a living agent, the force of moving wind or water, the attractive or repulsive force of a magnet,

etc.), the latter only having power to modify or prevent motion (such as friction, resistances of all kinds, or the tension of a string tied to a fixed point). If a book is let fall upon a table, the resistance of the table at first changes its state of motion into one of rest, yet continues to act after the state of rest has been produced, neither changing nor tending to change that state, but only *preventing* motion. These *passive* forces are called into play by one or more of the *active* sort, and their numerical magnitude depends chiefly on the magnitude of these latter. They are therefore generally spoken of as **reactions.**

161. **Equal Forces.**—Dynamically, two forces are said to be equal if, when they are applied to the *same* body for *equal* intervals of time, they tend to impart the same velocity, or change of velocity, to the body.

Statically, two forces are said to be equal if, when they are applied to a particle in opposite directions, the particle remains at rest.

That one of these definitions leads to the other can be seen by the following considerations: If two equal forces act on the same particle in exactly opposite directions, the motion which one force tends to impart is exactly the reverse of that which the other tends to impart. The particle cannot move in opposite directions at the same time, and there is no reason why it should move in one direction rather than the other. Hence it will remain at rest, and the two forces will be said to *balance*, or be *in equilibrium.*

162. **Systems of Forces.**—The forces with which we shall deal in *Statics* will always be supposed to be kept in equilibrium. But it is often necessary to consider the properties of some of the forces apart from the rest. Any number of forces may be called a system.

Defining the resultant as in Dynamics (§ 149), we see that, when a number of forces have a resultant, we may reduce them to a system in equilibrium by applying an additional force equal and opposite to that resultant. For the whole system is then equivalent to two equal and opposite forces

(viz. the original resultant and the added force), and is therefore in equilibrium. This additional force is sometimes spoken of as the equilibrant.

Conversely, when any number of forces are in equilibrium, each force is equal and opposite to the resultant of all the rest taken together.

This theorem is of frequent application, and leads, moreover, to the following (alternative) *statical* definition :—

DEFINITION.—*When a system of forces can be balanced by a single force, a force equal and opposite to this is called their* **resultant.**

[A system of forces does not *necessarily* have a single resultant force. except when they all act on the same particle, as in the proof of § 149.]

163. Point of Application of a Force.—The conditions of equilibrium of a system of forces do not depend on the nature and *mass* of the body on which they act, but only on the forces themselves. We may, therefore, speak of a force as **acting at a point**, meaning a force "applied to any particle placed at that point." And the **point of application** of the force is "the point at which the particle acted on by the force is situated."

In Chap. XVIII., however, we shall see that the point of application is rather taken to signify the point of intersection of the lines of action of two or more forces.

164. Statical Units of Force.—The forces which occur most frequently in Statics are those due to *weight*. Hence the most convenient statical unit of force is the *weight of a pound*, and this we call "a force of 1 lb."

If the French system of weights and measures is used, the statical units of force will be the weight of a gramme or of a kilogramme (1,000 gramme).

165. Forces may be Represented, Statically, by Straight Lines.—In order to define a force, statically, it is necessary to specify

(i) its point of application,
(ii) its direction,
(iii) its magnitude.

All these data will be specified by a straight line of finite length, provided that

(i) the line is drawn from the *point of application* of the force,

(ii) it is drawn pointing in the *direction* of the force,

(iii) its *length* is proportional to the *magnitude* of the force.

Such a straight line is said to *represent* the force.

The *sense* of the direction may be shown by an arrow drawn on or by the side of the line, or by the *order* of the letters used in naming the line. Thus **AB** represents a force acting from **A** towards **B**, **BA** a force acting from **B** towards **A**. (Compare §§ 146, 148.)

166. On the Choice of a Scale of Representation.—In order that the length of a straight line may represent the magnitude of a force, the line should properly contain as many units of length as the force contains units of force. Thus, if a line 1 inch long represents a force of 1 lb., a line 2 inches long will represent a force of 2 lb., and so on. Very often, however, it is necessary to adopt some other scale of representation suggested by the conditions of the problem. *We may so choose the scale of representation that* one *of the forces is represented by a straight line of any length we please.* When this has been done, a line of double that length will represent double that force, and so on, so that the lines representing all *other* forces will be fully determined.

It will often be necessary to represent a force *in magnitude and direction only* by a straight line *not* drawn from its point of application. A force will be represented to this extent by *any straight line drawn equal and parallel* to the line which fully represents it, for *parallel straight lines are to be regarded as having the same direction.*

167. THE PARALLELOGRAM OF FORCES.—**If two forces acting on the same particle be represented by two adjacent sides of a parallelogram, drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point.**

This has already been fully discussed in Dynamics, § 150, but the following treatment of the proposition is of great interest, being based on Newton's original proof, modified so as to apply to constant forces.

Let two constant forces P, Q be applied to a particle at rest at **A**, in directions **AB**, **AD**, respectively.

Let **AB** be the distance the particle would traverse in a given time t if it were set in motion by the constant force P alone.

Let **AD** be the distance traversed in the same time if acted on similarly by Q alone.

Complete the parallelogram **ABCD**, and join **AC**.

Then shall **AC** be the distance traversed by the particle in the time t if set in motion by both forces P, Q acting on it simultaneously in directions parallel to **AB**, **AD**, respectively.

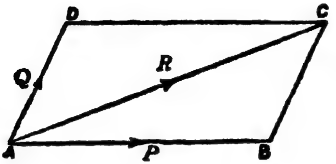

Fig. 40.

Also **AB**, **AD** shall represent the *forces* P, Q, and **AC** shall represent their resultant on the same scale.

(i) For, since the force Q always acts parallel to the line **BC**, it can have no effect in changing the rate at which the particle approaches **BC** in consequence of the other force P. Therefore the particle will reach the line **BC** in the same time, whether the force Q be applied or not. Therefore at the end of the time t it must be *somewhere* in the line **BC**.

Similarly it must be somewhere in **DC**.

Therefore it must be at **C**.

(ii) Hence the resultant of P and Q must be that force which would move the particle from rest along **AC** in the time t. It must therefore act in the direction **AC**.

And since the distance traversed in the given time t by a particle starting from rest is proportional to the force acting on it (§§ 36, 68), therefore **AB**, **AD**, **AC** are proportional to P, Q and their resultant.

That is, **AB** represents P, **AD** represents Q, and **AC** represents the resultant of P and Q, all on the same scale. [Q.E.D.]

168. Experimental Verification of the Parallelogram of Forces.

(*a*) Mechanical Details.—Take three strings; knot them together in a point. To their ends attach any three weights

P, Q, R, say P, Q, R lb. respectively (any two of which are together greater than the third). Allow one string to hang freely with its suspended weight R, and pass the other two over two smooth pulleys H, K, fixed in front of a vertical wall (Fig. 41).

(*b*) GEOMETRICAL CONSTRUCTION. — When the strings have taken up a position of equilibrium with the knot at A, measure off on the *wall* (not on the string) lengths AB, AD, containing P and Q units of length along AH, AK respectively. Complete the parallelogram ABCD, and join AC.

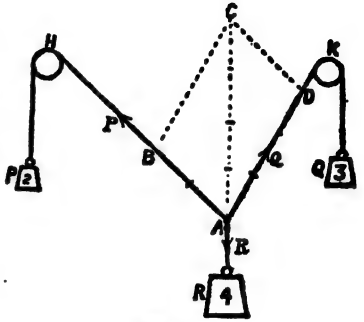

Fig. 41.

(*c*) OBSERVED FACTS.— *Then it will be invariably found*

(i) *that the diagonal* AC *is vertical,*
(ii) *that* AC *contains R units of length.*

(*d*) DEDUCTIONS.—Now the knot A is in equilibrium under the pulls P, Q, R acting along the strings respectively. Therefore the resultant of P, Q is equal and opposite to the weight R, that is, a force R, acting vertically upwards;

∴ AC represents the resultant in magnitude and direction.

Therefore the resultant is represented by the diagonal of the parallelogram whose sides represent the two forces. [Q.E.D.]

Fig. 41 is drawn for the case in which $P = 2$ lb., $Q = 3$ lb., $R = 4$ lb. The measured lengths AB, AD must therefore contain 2 and 3 units of length respectively. When the parallelogram is constructed, the diagonal AC will be found to be vertical and to contain 4 units of length. *See also* § 325, *Exp.* 14, 15.

Example.—To find graphically the resultant of forces of 7 lb. and 11 lb., whose directions include an angle of 60°.

Take any unit of length and measure off AB, AD containing 7 and 11 units respectively, making $\angle BAD = 60°$. Complete the parallelogram ABCD. (Fig. 42.)

Then **AC** represents the resultant.

On **AC** mark off from **A** a scale of the selected units. Then **C** will be found to lie between the 15th and 16th marks, so that **AC** contains about $15\frac{3}{4}$ units.

Therefore the resultant force = $15\frac{3}{4}$ lb. wt. roughly.

169. Half the Parallelogram is Sufficient.—Since **BC** is equal and parallel to **AD**, it represents the force Q in magnitude and direction (though not in position; see § 166). Hence, if two forces acting on a particle are represented in magnitude and direction by two sides of a triangle, **AB, BC**, their resultant is represented in magnitude and direction by the third side **AC**, its point of application being that of the forces. This leads to the following theorem.

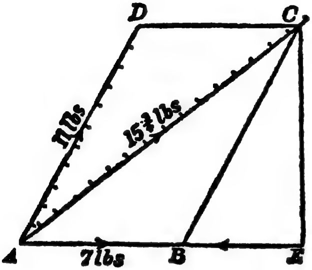

Fig. 42.

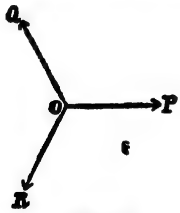

Fig. 43.

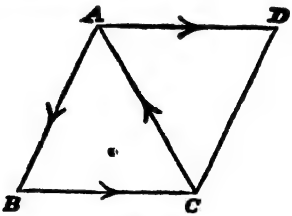

Fig. 44.

170. The Triangle of Forces.—*If three forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle taken in order,* they shall be in equilibrium.*

* The sides of a triangle or polygon are said to be *taken in order* when of any two adjacent sides one is drawn towards and the other away from their common angular point (§ 130).

Let three forces P, Q, R, acting on the same particle at **O**, be represented in magnitude and direction (but not in position) by the sides **BC**, **CA**, **AB** of the triangle **ABC**, respectively.

Then shall P, Q, R be in equilibrium.

Complete the parallelogram **ABCD**, so that **AD** is equal and parallel to **BC**.

The forces R, P are represented in magnitude and direction by **AB**, **AD**, and they act at **O**. Hence, by the Parallelogram of Forces, their resultant is similarly represented by **AC**, and also acts at **O**.

But Q is represented by **CA**.

Therefore the resultant of R, P is equal and opposite to the third force Q.

Therefore the three forces are in equilibrium. [Q.E.D].

171. **The angle between any two forces is the supplement of the corresponding interior angle of the triangle.**

Thus $\angle$ between R, P (*i.e.* $\angle$ **ROP** in Fig. 43) $= \angle$ **BAD** $= 180° - \angle$ **ABC**.

OBSERVATION.—The three forces *must not act along the sides of the triangle.* They must act *at a point* in directions parallel to these sides, as in Fig. 43; otherwise they cannot be applied to the same particle, and the proof fails.

172. Converse of the Triangle of Forces.

If three forces acting on a particle are in equilibrium, any triangle whose sides are *parallel* to the directions of the forces shall have the lengths of these sides *proportional* to the magnitudes of the forces.

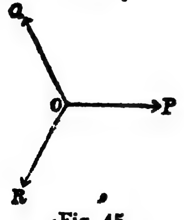

Fig. 45.

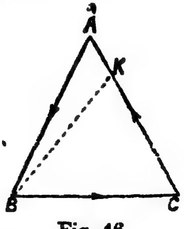

Fig. 46.

Let P, Q, R be three forces in equilibrium acting at **O**, and let **ABC** be any triangle whose sides **BC**, **CA**, **AB** are parallel to P, Q, R.

Then, if the scale of representation be properly chosen, **BC**, **CA**, **AB** shall represent P, Q, R in magnitude as well as direction.

For let the length **BC** be chosen to represent P. (§ 166). If **CA** does *not* represent Q, let **CK** represent Q.

Then, by the Parallelogram or Triangle of Forces, the resultant of P and Q is represented by **BK**. But P, Q, R are in equilibrium. Hence R must be represented by **KB**, equal and opposite to **BK**. But this is contrary to hypothesis, since R acts in the direction **AB**. Therefore Q cannot be represented in magnitude by any other length than **CA**, and therefore also, by the above reasoning, R is represented by **AB**.

OBSERVATION.—The above conditions may be expressed by the relations $\frac{P}{Q}=\frac{\mathbf{BC}}{\mathbf{CA}}$, $\frac{Q}{R}=\frac{\mathbf{CA}}{\mathbf{AB}}$, $\frac{R}{P}=\frac{\mathbf{AB}}{\mathbf{BC}}$, which may also be written $\frac{P}{\mathbf{BC}}=\frac{Q}{\mathbf{CA}}=\frac{R}{\mathbf{AB}}$.

From similar triangles (see Appendix, § 8), any triangle whose angles are equal to those of **ABC** has its sides proportional to those of **ABC**. Hence, if *any* triangle is drawn whose sides are parallel to **BC**, **CA**, **AB**, these sides also will represent P, Q, R, but on a different scale.

173. The Polygon of Forces.—*If any number of forces acting on the same particle can be represented in magnitude and direction by the sides of a closed polygon taken in order, they shall be in equilibrium.*

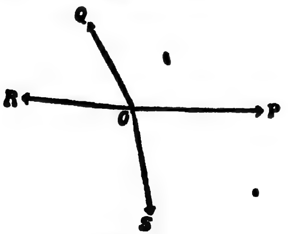

Fig. 47.

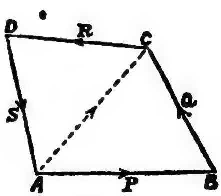

Fig. 48.

Let the forces P, Q, R, S, acting on a particle at **O**, be represented in magnitude and direction (but not in position) by the sides **AB**, **BC**, **CD**, **DA** of a polygon **ABCD**.

Then shall the forces be in equilibrium.

For, as in the Triangle of Forces, or § 169, the resultant of the forces P, Q is represented in magnitude and direction by **AC**.

Therefore the resultant of P, Q, R is the same in magnitude and direction as that of forces **AC** and **CD**, and is similarly represented by **AD**.

But the force S is represented by **DA**.

Therefore S is equal and opposite to the resultant of P, Q, R.

Therefore the forces P, Q, R, S are in equilibrium. [Q.E.D.]

The properties stated in § 171 are equally applicable to the Polygon of Forces.

We have considered the case of four forces, but the proof may be similarly extended to any number of forces.

174. To construct the resultant of any number of forces acting on a particle.

Let the given forces be represented by the straight lines **AB**, **BC**, **CD**, taken in order, forming all the sides but one of a polygon. Then, if the polygon be completed by drawing the remaining side *from* **A**, the extremity of the first side, *to* **D**, the extremity of the last side, the line **AD** will represent the resultant force.

This is evident from the last article.

175. It is immaterial in what order the forces are represented. The form of the polygon will depend on which force is represented first, which next, and so on; but the line representing the resultant will be the same in every case.

[For, consider the case of two forces P, Q, acting at **A** (Fig. 40). If we represent P first and Q second, the lines representing them will be **AB**, **BC** respectively, and the resultant will be represented by **AC**. If we represent Q first and P second, the lines representing them will be the opposite sides **AD**, **DC** respectively, and therefore the resultant will still be represented by **AC**. The same property may be extended to any number of forces by interchanging their order of successive representa tion, taking two at a time.]

176. Converse of the Polygon of Forces.

If any number of forces acting on a particle are in equilibrium, a closed polygon can be drawn whose sides represent these forces both in magnitude and direction.

Let the forces P, Q, R, S, acting at **O**, be represented by **AB**, **BC**, **CD**, **DE** respectively, these lines being placed end to

end. Then, if the figure **ABCDE** be not a closed polygon, the forces will have a resultant represented by **AE** (§ 174), and will, therefore, not be in equilibrium.

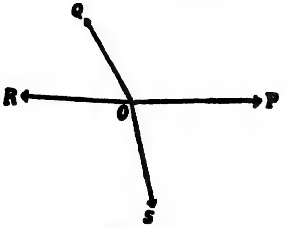

Fig. 49.

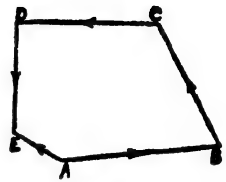

Fig. 50.

Therefore the lines representing forces in equilibrium must form a closed polygon. (*See also* § 325, *Exp.* 16.)

NOTE. *The converse of the Polygon of Forces is less complete than that of the Triangle.*—That is, it is not true that the sides of any polygon respectively *parallel* to the forces will be necessarily *proportional* to them.

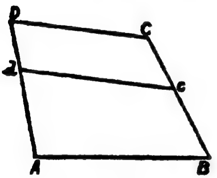

Fig. 51.

For, if there are more than three forces, we can draw any number of different polygons, such as **ABCD**, **ABcd** (Fig. 51), each having its sides parallel to the directions of the forces, and the sides of one polygon are not necessarily proportional to those of another. The sides of each polygon represent *a* system of forces in equilibrium acting in these directions, but not necessarily *the* system we started with. We cannot, therefore, except in the case of three forces, determine the ratios of the forces from merely knowing their directions.

177. Applications of the Parallelogram, Triangle, and Polygon of Forces.

(1) *The resultant of two equal forces bisects the angle between them.*

This follows from the fact that the Parallelogram becomes a rhombus, in which it is known that the diagonals bisect the angles. It also follows from first principles, through considerations of symmetry.

(2) *If any three forces are in equilibrium, any two of the forces are together not less than the third.* For, in the Triangle

of Forces, any two sides are together greater than the third (Euc. I. 20).

If two forces are together *equal* to the third, they will balance if the first two forces act in the same straight line and in the opposite sense to the third.

(3) *The resultant of two forces P, Q is greatest* when both act in the same direction, and is then $P+Q$. It is *least* when they act in opposite senses in the same straight line, and is then either $P-Q$ acting in the direction of P or $Q-P$ in the direction of Q.

This is an easy inference from (2).

(4) *If three equal forces are in equilibrium, the angle between any two of them is* 120°. For the Triangle of Forces is equilateral; therefore each of its angles is 60°, and the angle between the corresponding pair of forces

$$= 180^\circ - 60^\circ = 120^\circ.$$

It follows also, from considerations of symmetry, that if *any number* of equal forces act on a particle at equal angles, all round, so as to form a symmetrical star, they will be in equilibrium.

(5) *The Perpendicular Triangle of Forces.*—If three forces proportional to the sides of a triangle act on a particle in directions perpendicular to these sides taken in order, they shall be in equilibrium.

For if the triangle **ABC** be turned through a right angle into the position **DEF**, its sides, taken in order, will be brought parallel to the forces, and will therefore represent them both in magnitude and direction. Therefore the forces will be in equilibrium.

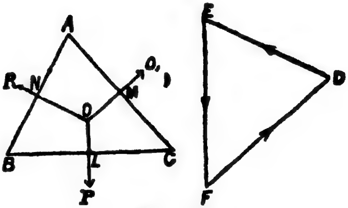

Fig. 52.

Conversely, if three forces acting perpendicularly to the sides of a triangle keep a particle in equilibrium, they shall be proportional to these sides.*

* This theorem has an important application to Hydrostatics, in proving that the pressure at a point in a fluid is the same in all directions.

(6) *If two forces be represented by the sides of a triangle* ***both drawn from*** *their point of application, their resultant is represented by twice the bisector of the base drawn from that point.*

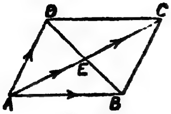

Fig. 53.

For let **AB**, **AD** represent the forces. Complete the parallelogram of forces **ABCD**; then **AC** represents the resultant. But the diagonals of a parallelogram bisect each other. Hence **AC** bisects **BD** in **E**, and **AC** is twice the bisector **AE**.

This result affords an easy construction for the resultant of two forces acting *from* a point. Simply join **BD**, and bisect it at **E**. Then the resultant is represented by twice the bisector **AE**, *i.e.* by twice the *medium line* (as it is called).

(7) *To find where a particle* **O** *must be placed inside a triangle* **ABC** *so that it may be in equilibrium under forces to the vertices represented by* **OA**, **OB**, **OC**.

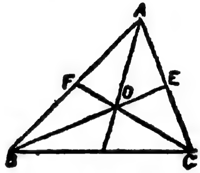

Fig. 54.

Let **D** be the middle point of **BC**. Then, by the last proposition, the forces **OB**, **OC** have a resultant **2OD** along **OD**. This must be equal and opposite to the third force **OA**. Hence **O** must lie on the bisector **AD** at a point such that

$$OA = 2DO.$$

$$\therefore \quad DA = 3DO \quad \text{and} \quad DO = \tfrac{1}{3}DA.$$

Similarly, **O** must lie on each of the other bisectors, or medians. We thus have a mechanical proof of the geometrical theorem that the three medians of a triangle meet at one point, which is a point of trisection in each of them.

[The point **O** is the *centre of gravity* of the triangle; see Chap. XXV.]

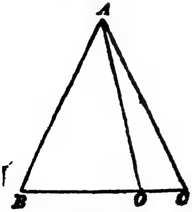

Fig. 55.

Examples.—(1) Two forces act along the sides **AB**, **AC** of a triangle, and are represented in magnitude by **AB** and three times **AC** respectively. To find where their resultant cuts the base, and to determine its magnitude.

Let the resultant cut the base in **O**. Then, by the Triangle of Forces, the force **AB** is equivalent to forces **AO** along **AO**, **OB** at **A** parallel to **OB**.

Similarly the force 3AC is equivalent to forces 3AO along AO, 3OC at A parallel to OC.

Therefore the two given forces are equivalent to forces of magnitude and direction 4AO, OB, and 3OC.

If the resultant acts along AO, the two latter forces must balance.

$$\therefore \quad OB = 3OC, \quad \text{and} \quad BC = 4OC;$$

$$\therefore \quad OC = \tfrac{1}{4}BC, \quad \text{and} \quad BO = \tfrac{3}{4}BC.$$

Also the resultant is the remaining force, viz. 4AO acting along AO.

The above is a strict treatment of the question. If we remember that all the forces really act at A, we may speak more freely, as thus: "The force AB is equivalent to forces AO and OB," and so on.

(2) ABC is any triangle, and E is the middle point of BC. To find the resultant of forces at A represented in magnitude, direction, and sense by 2BA, 2AE, and AC respectively.

By the Triangle of Forces, the resultant of forces 2BA, 2AE is a force 2BE parallel to BC. But 2BE = BC; therefore the three forces are equivalent to forces BC, AC acting at A.

Produce BC to F, so that CF = BC.

Then the two last forces are represented by CF, AC respectively, and they act at A.

Therefore the required resultant is represented by AF.

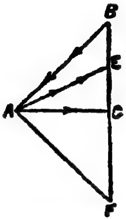

Fig. 56.

178. To find the magnitude of the resultant of two forces P, Q in directions at right angles to one another.

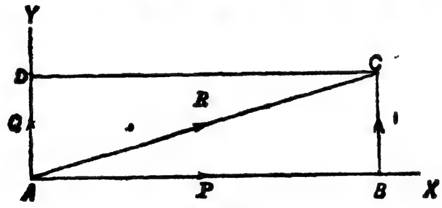

Fig. 57.

Let AB, AD represent the two forces P, Q.

Complete the rectangle ABCD.

Then AC represents the resultant force.

Let this be $= R$. Then, by Euclid I. 47,

$$AC^2 = AB^2 + BC^2 = AB^2 + AD^2;$$

$$\therefore \quad R^2 = P^2 + Q^2 \quad \text{.......................} \quad (1)$$

or $\qquad$ resultant force $R = \sqrt{(P^2 + Q^2)}$.

Example.—To find the resultant of forces of 2 lb., 2 lb., and 1 lb. at a point, the angle between the first and second being 90°, and that between the second and third being 45°.

The resultant of the first and second forces is easily seen to be $2\sqrt{2}$ lb., and, as it bisects the angle between them, it is itself at right angles to the third force. Hence, if R be the resultant of the three forces, $R^2 = (2\sqrt{2})^2 + 1^2 = 8 + 1 = 9$, and $R = 3$ lb.

EXAMPLES XV.

1. Draw a diagram, as well as you can to scale, showing the resultant of two forces, equal to the weights of 5 lb. and 8 lb., acting on a particle, with an angle of 60° between them; and, by measuring the resultant, find its numerical value.

2. A string supports a mass of 3 lb. at its extremity, another mass of 5 lb. above the first, and a third of 7 lb. above the second. Find the tensions of the three parts of the string.

3. When two forces act together in the same straight line they have a resultant of 18 lb., and when they act in opposite directions their resultant is 2 lb. Find the two forces.

4. Two forces act on a particle, and their greatest and least resultants are 20 lb. and 4 lb. respectively. Find the forces.

5. The greatest resultant which three given forces acting at a point can have is 36 lb., and the least is 0. Find the greatest and least possible values of the greatest of the three forces.

6. **ABC** is a triangle. Find the resultant of forces acting at **A** which are represented in magnitude, direction, and sense by

(i) 2**AB** and 7**AC**; (iii) 3**BA** and **AC**; (v) 2**AB**, 3**AC**, 4**BC**;
(ii) 2**BA** and **CA**; (iv) **AB** and 4**CA**; (vi) 2**BA**, 5**AC**, 3**CB**.

7. Show that the three forces which are represented in magnitude and direction by the straight lines drawn from the angular points of a triangle to the middle points of the opposite sides are in equilibrium.

8. Seven forces act at a point so that the angle between every consecutive two is 45°. The first force and the alternate forces from the first are each 4 lb.; the second force and those alternate from it are each 10 lb. Find the resultant of the whole.

9. In the side **AB** of the triangle **ABC** a point **D** is taken so that **AD** : **DB** = 14 : 11. Forces act at **C** in the directions **CA**, **CB**, **CD** proportional to **CA**, 2**CB**, and **CD** respectively. Find the magnitude and direction of their resultant.

10. Two forces act at a point at right angles to each other. Find the resultant when the forces are

(i) 15 lb. and 20 lb.; (iii) 119 lb. and 120 lb.;
(ii) 40 lb. and 42 lb.; (iv) 240 lb. and 100 lb.

11. If the resultant of two forces acting at a point at right angles to each other be 53 lb., and one of the forces be 28 lb., find the other force.

12. Two forces whose magnitudes are as 5 to 12, acting on a particle at right angles to each other, have a resultant of 78 lb. Find the forces.

13. Two forces acting in opposite directions to one another on a particle have a resultant of 34 lb., and if they acted at right angles their resultant would be 50 lb. Find the two forces.

14. Two forces act on a particle at right angles to each other, and the greater of the two forces is 32 lb. less than the resultant, which is 305 lb. Find the smaller force.

15. The smaller of two forces, which act on a particle at right angles to each other, is 8 lb., and the sum of the resultant and the larger force is 288 lb. Find the resultant and the larger force.

16. Forces represented in magnitude and direction by AB, AC, AD, AE, AF act at the corner A of a regular hexagon ABCDEF. Find the magnitude and direction of their resultant.

17. ABDC is a parallelogram, and AB is bisected at E. Show that the resultant of the forces represented in magnitude and direction by AD and AC is double the resultant of the forces represented in magnitude and direction by AC and AE, and acts in the same direction.

18. The sides AB, AC of a triangle ABC are bisected in F and E. Find the resultant of forces represented in magnitude and direction by CF and BE.

19. Forces represented in magnitude and direction by the diagonals AC, BD of a parallelogram ABCD act at a point O. Find the magnitude and direction of their resultant.

20. A tree is pulled north, south, east, and west by four ropes, and the forces of tension in the ropes are equal to 35, 60, 20, and 80 lb. respectively. Find the magnitude of the resultant force.

21. Can three forces whose magnitudes are in the proportion of 4, 7, and 11 keep a particle at rest?

22. Show that the resultant of two perpendicular forces $P+Q$ and $P-Q$ is equal in magnitude to the resultant of two perpendicular forces $\sqrt{2}P$ and $\sqrt{2}Q$.

CHAPTER XVI.

RESOLUTION OF FORCES.

179. **Definitions.**—The process of replacing a single force by two or more forces having that force for their resultant is known as the **resolution** of forces, and is the reverse process to the *composition* of forces.

The several forces are called the **components** of the given force.

Thus we speak of *resolving* a force into *components*, and of *compounding* two or more forces into a single *resultant*.

To resolve a force into components in two given directions **AB**, **AD**, it is only necessary to draw the straight line **AC** representing the given force, and to draw **CD**, **CB** through **C** parallel to **BA**, **DA** respectively. Then, by the Parallelogram of Forces, **AB**, **AD** will represent the required components.

180. **Resolution at Right Angles.**—It is often necessary to resolve a force into components along and perpendicular to a given line, the cases most frequently occurring in elementary problems being those where the force makes angles of 30°, 45°, or 60° with the given line.

Let **AC** represent the given force P, and **AB** be the given line, and let $\angle$**BAC** $= A$.

Draw **AD** perpendicular to **AB**, and complete the parallelogram **ABCD**. Then **AB**, **AD** represent the two required components. Let **X**, **Y** denote these components respectively.

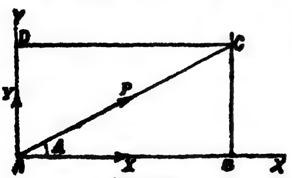

Fig. 58.

By referring to § 135, or to Appendix, §§ 5, 6, or from trigonometry,* the student will have no difficulty in verifying the following very important results :—

Where the angle A =	30°	45°	60°
The component X =	$\frac{1}{2}\sqrt{3}.P$	$\frac{1}{2}\sqrt{2}.P$	$\frac{1}{2}P$
The component Y =	$\frac{1}{2}P$	$\frac{1}{2}\sqrt{2}.P$	$\frac{1}{2}\sqrt{3}.P$

181. Definition.—**The resolved part or resolute of a force along a given straight line is the component of the force along that line** ***when the other component is perpendicular to it.***

Thus, let **AB** be the given straight line, and let the given force P be represented by **AC.** Drop **CM** perpendicular on **AB.**

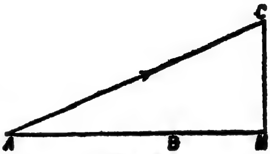

Fig. 59.

Then **AM, MC** represent the components of P along and perpendicular to **AB.**

Therefore **AM** represents the *resolved part of P along* **AB.**

This is the X-component of § 180; the Y-component is called the resolved part perpendicular to the line.

If P is *along* or *parallel* to the line, $X = P$.

If P is *perpendicular* to the line, $X = 0$.

If the angle is obtuse, the resolved part is considered negative, as in Example 1, below. Hence the following results :—

If the inclination of the force P is	0°	30°	45°	60°	90°	120°	135°	150°	180°
its resolved part along the line is P multiplied by	$+\frac{\sqrt{4}}{2}$	$+\frac{\sqrt{3}}{2}$	$+\frac{\sqrt{2}}{2}$	$+\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$	$-\frac{\sqrt{1}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{4}}{2}$

* Those who are acquainted with the rudiments of Trigonometry will see that $X = P \cos A$ and $Y = P \sin A$, whence these results would follow at once.

Examples.—(1) A force of 10 lb. makes an angle of 135° with a given line. To resolve it into components along and perpendicular to that line.

Let XAC = 135°, and let AC represent 10 lb.

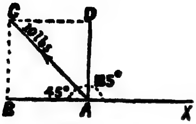

Fig. 60.

Produce XA to B, and complete the rectangle ABCD.

Then AB, AD represent the required components.

Now $\angle BAC = 180° - 135° = 45°$.

Hence the components along AB and AD are

$$AB = AC\sqrt{\tfrac{1}{2}} = 10 \times \tfrac{1}{2}\sqrt{2} = 5\sqrt{2} \text{ lb.},$$
$$AD = AC\sqrt{\tfrac{1}{2}} = 10 \times \tfrac{1}{2}\sqrt{2} = 5\sqrt{2} \text{ lb.}$$

But the force $5\sqrt{2}$ lb. along AB acts in the *opposite* direction to that in which AX is drawn; hence this force is to be regarded as a *minus quantity.*

Therefore the required components are

$-5\sqrt{2}$ lb. along AX, $\quad +5\sqrt{2}$ lb. perpendicular to AX.

(2) Three forces of 5 lb., 6 lb., and 4 lb. are inclined to one another at angles of 120°. To replace them by two forces acting along and perpendicular to the force of 5 lb., and to find the resultant of the three.

Let OP, OQ, OR represent the three forces. Produce PO to B, and draw DOE perpendicular to OP.

Then BOQ = BOR = 60°.

Therefore the force 6 lb. along OQ is equivalent to

$6 \times \frac{1}{2}$ lb. along OB

and $6 \times \frac{1}{2}\sqrt{3}$ lb. along OD.

Similarly, the force 4 lb. along OR is equivalent to

$4 \times \frac{1}{2}$ lb. along OB

and $4 \times \frac{1}{2}\sqrt{3}$ lb. along OE.

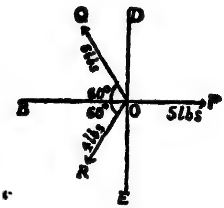

Fig. 61.

We also have the force 5 lb. along OP.

Therefore the three forces are equivalent to

$5 - 3 - 2$ lb. along OP and $3\sqrt{3} - 2\sqrt{3}$ lb. along OD,

i.e. zero along OB and $\sqrt{3}$ lb. along OD.

Therefore the required resultant acts along OD, and its magnitude

$= \sqrt{3}$ lb. $= 1{\cdot}732$ lb. nearly.

182. The algebraic sum of the resolutes of two forces in any direction is equal to the resolute of their resultant in the same direction.

Let AX be the given direction, and let the forces P, Q and their resultant R be represented as in Fig. 62.

Draw the perpendiculars BL, CM, DN on AX.

Then AL, AN, AM represent the resolutes of P, Q, R.

Now it is easy to prove geometrically* that AN = LM.

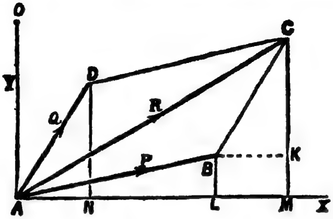

Fig. 62.

Therefore AL + AN = AL + LM = AM;

i.e. sum of resolutes of P and Q = resolute of R.

COR. The algebraic sum of the resolutes of *any number of forces* on a particle is equal to the resolute of their resultant in the same direction.

This follows at once by applying the theorem successively to the resultant of two of the forces, that compounded of this resultant with a third force, and so on.

Examples.—(1) To find the resultant of two forces of 1 lb. and 3 lb. inclined at an angle of 30°.

Resolve the second force into components along and perpendicular to the first. These components are $3\times\frac{1}{2}\sqrt{3}$ and $3\times\frac{1}{2}$ lb. respectively.

Hence the two forces are together equivalent to forces $1+\frac{3}{2}\sqrt{3}$ lb. and $\frac{3}{2}$ lb. acting along and perpendicular to the first force.

Let their resultant be R. Then, by § 178,

$$R^2 = (1+\tfrac{3}{2}\sqrt{3})^2 + (\tfrac{3}{2})^2 = 1 + 3\sqrt{3} + \tfrac{27}{4} + \tfrac{9}{4} = 10 + 3\sqrt{3};$$

$$\therefore\ R = \sqrt{(10+3\sqrt{3})} = \sqrt{(10 + 3\times 1{\cdot}732)} = \sqrt{15{\cdot}196};$$

$$\therefore\ \text{the resultant} = 3{\cdot}898 \text{ lb. nearly.}$$

(2) Two forces of $\sqrt{3}$ lb. and 1 lb. include an angle of 150°. To find the magnitude and direction of their resultant.

---

* For, drawing BK parallel to AX, the triangles DAN, CBK are equal in every respect; $\therefore$ AN = BK = LM.

Reduce the forces to two forces X, Y acting along and perpendicular to the direction of the force of $\sqrt{3}$ lb. Then, as in § 180,

$$X = \sqrt{3} - \tfrac{1}{2}\sqrt{3}.1 = \tfrac{1}{2}\sqrt{3} \text{ lb.},$$
$$Y = \tfrac{1}{2} \text{ lb.};$$
$$\therefore \quad R^2 = X^2 + Y^2 = \tfrac{1}{4} + \tfrac{3}{4} = 1;$$
$$\therefore \quad \text{resultant } R = 1 \text{ lb.}$$

Now X, Y, R can be represented as the sides of a semi-equilateral triangle;

$\therefore$ the resultant makes an angle 30° with the force of $\sqrt{3}$ lb.

183. To find the magnitude of the resultant of two forces inclined to each other at a given angle.

Let **AB**, **AD** represent the components P, Q.

Let X be the resolved part of Q along **AB**.

Complete the parallelogram **ABCD**. Then **AC** represents the resultant R.

Drop **CM**, **DN** perpendicular on **AB**.

Then **BM** = **AN** = X.

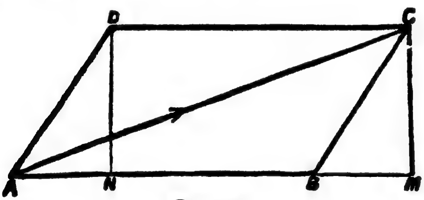

Fig. 63.

By Euc. II. 12, $AC^2 = AB^2 + BC^2 + 2AB \,.\, BM$.

Therefore $$R^2 = P^2 + Q^2 + 2PX \quad\text{.................. (1).}$$

If $\angle$**BAD** is obtuse, $\angle$**ABC** is acute, and X is negative, and the same thing follows from Euc. II. 13.

Hence, to calculate the resultant of any two forces, it is only necessary to determine X, the resolved part of one of the forces along the line of action of the other, and to substitute in the formula $R^2 = P^2 + Q^2 + 2PX$.

Alternative Method.—The triangle **DAN** or **CBM** being right-angled, we find the values (X, Y) of the resolutes **BM** and **CM** from that of **BC** or **AD**. Then, by Euc. I. 47, we can find **AC**, the hypotenuse of the right-angled triangle **AMC**, which represents R.

Examples.—(1) To find the resultant of forces of 7 lb. and 11 lb. inclined at an angle of 60°.

The resolved part of the second force in the direction of the first $= 5\frac{1}{2}$ lb.; therefore, if the resultant contains R lb.,

$$R^2 = 7^2 + 11^2 + 2.7.5\tfrac{1}{2}$$
$$= 49 + 121 + 77 = 247,$$

whence $$R = \sqrt{247} = 15{\cdot}716 \text{ lb. wt. } (cf. \S\ 168, \text{ Ex.}).$$

(2) To find the resultant when the same forces include an angle of 120°.

If the direction of the 7 lb. force is produced backwards, the 11 lb. force will be found to make an angle 60° with it.

Hence $$X = -5\tfrac{1}{2},$$
and $$R^2 = 7^2 + 11^2 + 2.7.(-5\tfrac{1}{2})$$
$$= 49 + 121 - 77 = 93,$$
whence $$R = \sqrt{93} = 9{\cdot}643 \text{ lb. wt.}$$

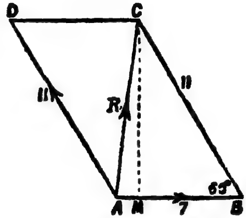

Fig. 64.

By the alternative method (see Fig. 64) since $BC = 11$ and the angle $CBM = 60°$, we have $BM = \frac{11}{2}$ and $CM = \frac{11}{2}\sqrt{3}$.

Hence $AC^2 = AM^2 + MC^2 = (7 - 5\frac{1}{2})^2 + (\frac{11}{2}\sqrt{3})^2 = \frac{9}{4} + \frac{363}{4} = 93$.

Thus $R = \sqrt{93}$ lb., as before.

184. Particular Cases.—For certain inclinations of the forces P, Q, we can apply (1) to write down the resultants in a convenient form :—*

For	0°,	$R^2 = P^2 + Q^2 + \sqrt{4}.PQ$;
	30°,	$R^2 = P^2 + Q^2 + \sqrt{3}.PQ$;
	45°,	$R^2 = P^2 + Q^2 + \sqrt{2}.PQ$;
	60°,	$R^2 = P^2 + Q^2 + \sqrt{1}.PQ$;
	90°,	$R^2 = P^2 + Q^2 + \sqrt{0}.PQ$;
	120°,	$R^2 = P^2 + Q^2 - \sqrt{1}.PQ$;
	135°,	$R^2 = P^2 + Q^2 - \sqrt{2}.PQ$;
	150°,	$R^2 = P^2 + Q^2 - \sqrt{3}.PQ$;
	180°,	$R^2 = P^2 + Q^2 - \sqrt{4}.PQ$.

In this table, the expressions for the resultant are given for all angles that are multiples of 15°, with the exception of 15°, 75°, 105°, 165°.

* These are particular cases of the general trigonometrical formula
$R^2 = P^2 + Q^2 + 2PQ \cos A$ for *any* angle A.

If the angle between the forces has any one of these four values, we must draw two perpendicular lines **OX**, **OY** inclined at angles of 45° to the direction of one of the forces, thus :—

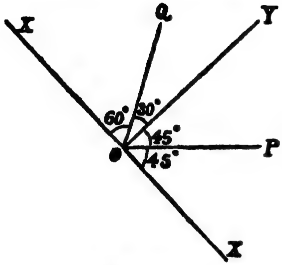

Fig. 65.

Example.—To find the resultant of forces 3 and 4 lb. inclined at an angle of 75°.

Take a line **OY** between the two forces, making angles 45° and 30° with them respectively.

Draw **OX** perpendicular to **OY**, and replace each force by its components along **OX**, **OY**. The work will now be easy, and is left as an exercise for the student. It will be found that the rectangular components of R are 1·328 and 5·426 lb., and that

$$R = 5{\cdot}586 \text{ lb.}$$

185. Conditions of Equilibrium.—In order that a system of forces acting on a particle in one plane may be in equilibrium, it is necessary and sufficient that the sums of the resolved parts of the forces along two straight lines at right angles shall be separately zero.

Let **OX**, **OY** be two straight lines at right angles.

If the forces have a resultant R, let X, Y be the resolved parts of this resultant along **OX**, **OY**. Then, by § 178,

$$R^2 = X^2 + Y^2;$$

and, by § 182, Cor.,

$$X = \text{sum of resolved parts of forces along } \mathbf{OX}.$$
$$Y = \quad ,, \quad ,, \quad ,, \quad ,, \quad \mathbf{OY}.$$

Now, *for equilibrium*, R must $= 0$. Hence $X^2 + Y^2$ must $= 0$. But square quantities, such as X^2 and Y^2, cannot be negative; hence they must severally be zero. Thus the condition $R = 0$ involves the pair of conditions $X = 0$ and $Y = 0$.

Conversely, if these two conditions are satisfied, $R = 0$, and the forces are in equilibrium.

OBSERVATIONS.—If X were zero and Y were not zero, the forces would have a resultant Y perpendicular to **OX**.

The proposition shows that, if the forces are in equilibrium, the sum of the resolved parts along *every* straight line is zero. But this will

necessarily be the case if the sums of their resolved parts along *two* perpendicular straight lines be zero.

The same thing is true if the two straight lines are not perpendicular. For, if the forces were not in equilibrium, their resultant would have to be perpendicular to both lines, which is impossible.

Examples.—(1) A particle is in equilibrium under the following system of horizontal forces*, viz. a force of P units to the north, Q to the west, 4 to the south-west, 3 to the east. Find P and Q.

Here $X = 3 - Q - 4 \times \frac{1}{2}\sqrt{2}$

$= 3 - 2\sqrt{2} - Q,$

and $Y = P - 2\sqrt{2}.$

Since the system is in equilibrium,

$$3 - 2\sqrt{2} - Q = 0,$$

and $$P - 2\sqrt{2} = 0.$$

Hence $P = 2\sqrt{2} = 2{\cdot}828$ units,

and $Q = 3 - 2\sqrt{2} = {\cdot}172$ unit.

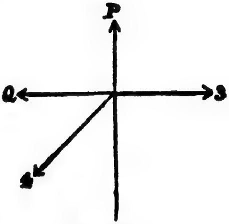

Fig. 66.

Note.—If the value of Q had turned out negative, this would indicate that Q must act in the opposite sense to that assumed.

(2) To indicate the forces which enable a ship to sail partly against the wind.

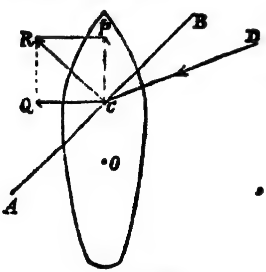

Fig. 67.

Let OCP be the direction of the ship's length, DC the direction of the relative velocity of wind, ACB the direction of the sail, which must lie within the angle DCP. Resolve the *velocity* of the wind into two components, one perpendicular to AB, the other along AB. The former component produces a pressure of the wind against the sail, while, since there is little friction, the effect of the latter component may be neglected. Hence the wind exerts a force R perpendicular to AB, represented by CR.

Now resolve the force R into its components P, Q along and perpendicular to the ship's length. The component P causes the ship to move forward. The component Q, acting broadside or athwart the ship, meets with very great resistance, and causes the ship to make *lee-way*, as it is called; but the amount of this is so small that the path of the ship is *very nearly* in the direction CP, and still makes an acute angle with CD.

* That is, their directions all lie in a horizontal *plane*.

In like manner, if a barge be drawn by a tow-rope pulled from the bank in the direction CR, it will move *very nearly* in the direction of its length CP.

[When the ship moves uniformly, the forces on it are of course in *equilibrium*, the components of R being balanced by the resistances of the water. The greater the velocity of the ship, the greater the resistance it encounters; hence the greater must be the motive force P.]

EXAMPLES XVI.

1. Resolve the following forces into components, along and perpendicular to the straight lines to which they are inclined at the given angles—

(i) 5 lb., 0°;	(iv) 12 oz., 60°;	(vii) 4 tons, 135°;
(ii) 8 lb., 30°;	(v) 5 cwt., 90°;	(viii) 24 grm., 150°;
(iii) $10\sqrt{2}$ tons, 45°;	(vi) 32 kilog., 120°;	(ix) 3 mgr., 180°.

2. A force equal to the weight of 20 lb., acting vertically upwards, is resolved into two forces, one of which is horizontal and equal to the weight of 10 lb. What is the magnitude and direction of the other component?

3. A vertical force of 12 lb. is resolved into two equal components, one of which makes an angle of 30° with the vertical. Find the magnitude and direction of the other.

4. The vertical resolved part of a force making an angle of 60° with the vertical is 5 lb. Find the force and its horizontal component.

5. A force of 4 lb. bisects the angle between two straight lines which include an angle of 120°. Find (i) the components, (ii) the resolved parts, of the force along these lines.

6. Find the magnitudes of the resultants of the following pairs of forces inclined at the given angles, namely—

(i) 8 and 10 oz., 0°;	(v) 17 and 144 mgr., 90°;
(ii) 2 and $\sqrt{3}$ tons, 30°;	(vi) 12 and 24 kilog., 120°;
(iii) 1 and $\sqrt{2}$ lb., 45°;	(vii) $5\sqrt{2}$ and 2 lb., 135°;
(iv) 3 and 6 grammes, 60°;	(viii) 7 and 8 cwt., 150°;
(ix) 5 and 4 lb., 180°.	

7. Two forces of 8 lb. and 12 lb. act at angles of 60°, 90°, 120° in succession. Compare the resultants in the three cases.

8. Two forces of 5 lb. and 12 lb. act at a point and are inclined to each other at an angle of 60°. Find the magnitude of their resultant.

9. Indicate two forces, at right angles to each other, which could maintain equilibrium with the forces of Ex. 8.

10. Two forces P and Q act at such an angle that their resultant is equal to P; show that if P be replaced by $2P$ the new resultant will be perpendicular to Q.

11. AB, AC represent two forces of 25 lb. and 15 lb. respectively. If CD be drawn perpendicular to AB, AD would on the same scale represent 3 lb. Find the resultant of the forces in AB, AC.

12. Show that the greater the angle between the lines of action of two forces acting at a point, the less will be their resultant.

13. A force of 12 lb., acting northwards, is resolved into three components, of which one is $6\sqrt{2}$ lb. north-westwards, and another 8 lb. westwards. Find the magnitude of the third component.

14. Find the magnitude of the resultant of three forces of 1, 2, and 3 lb. acting at a point along lines making angles of 120° with one another.

15. The sides AB, BC, CA of the triangle ABC are respectively 6, 10, and 8 inches long, and a force of 15 lb. acts at the point A in the direction of the line joining A to the middle point of BC. Find the forces which, acting along AB, AC, will produce the same effect as this force.

16. The following forces act on a particle: 4 lb. due east, $5\sqrt{2}$ lb. due north-east, $3\sqrt{2}$ lb. due south-east, and 7 lb. due south. Find the magnitude of their resultant.

17. Two forces, one of which is treble the other, act on a particle at right angles to one another, and are such that, if the smaller force be doubled and 8 lb. be added to the other, the direction of the resultant will be unchanged. Find the two forces.

18. Two equal forces act at a certain angle on a particle, and have a certain resultant. If the direction of one of the forces be reversed and its magnitude be doubled, the new resultant is of the same magnitude as before. Find the angle between the two equal forces.

19. Show that the resultant of the two equal resultants in Ex. 18 is equal to either of them.

*20. Forces of 33 lb. and 25 lb. act at a point and have for their resultant a force of 52 lb. Find the cosine of the angle between the two forces.

*21. Find the cosine of the angle between the directions of forces of 56 and 25 units which have a resultant of 39 units. Show that the angle itself exceeds a right angle.

22. A force of $12\sqrt{2}$ lb. acts along a straight line which is inclined at angles of 75° and 15° to two straight lines at right angles. Find the resolved parts of the force along these lines by first replacing it by its components along the internal and external bisectors of the angles between them.

23. Calculate, to two places of decimals, the magnitude of the resultant of forces of $4\sqrt{2}$ and 24 lb. when the angle between them is 15°, and when it is 105°.

24. Calculate, to two places of decimals, the magnitude of the resultant of forces of $\sqrt{2}$ and 5 lb. when the angle between them is 75°, and when it is 165°.

25. Show that, if forces of 25, 30, and 55 lb. keep a particle at rest, they must act in the same straight line.

26. Forces of 1, 2, 3, and 4 lb. respectively act along the straight lines drawn from the centre O of a square ABCD to the angular points A, B, C, D taken in order. Find their resultant.

27. Forces of 3 lb., 4 lb., 5 lb. act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find their resultant.

28. A boat is being towed by a rope making an angle of 30° with the boat's length; the resultant pressure of the water on the boat and rudder is inclined at 60° to the length of the boat, and the tension in the rope is equal to the weight of half a ton. Find the resultant force, supposing it to be in the direction of the boat's length.

EXAMINATION PAPER IX.

1. Explain carefully the manner in which forces can be geometrically represented.

2. What is meant by the *resultant* of two forces, and how can it be determined?

3. Enunciate the proposition known as the "Parallelogram of Forces," and describe an apparatus for verifying it experimentally.

4. Assuming the truth of the "Parallelogram of Forces," enunciate and prove the proposition known as the "Triangle of Forces."

5. The greatest resultant of two forces is 31 lb., and the least resultant is 17 lb.; what is the resultant when the two forces act at right angles to one another?

6. If five forces acting at a point can be represented in magnitude and direction by the sides of a pentagon, taken in order, show that they will be in equilibrium.

7. Forces of 5, 4, and 5 lb. act at a point O in directions parallel to the sides AB, BC, CA of an equilateral triangle respectively. Find their resultant.

8. Show that a force may be resolved into two components in any number of different ways, and explain what is meant by the *resolved part* of a force in any given direction.

9. Obtain the formula giving the magnitude of the resultant of any two forces acting at a point, in terms of the components and the resolved part of one component along the line of action of the other.

10. Two forces of 12 lb. and 15 lb. respectively, act at a point, and the angle between their lines of action is 60°. Find, to two places of decimals, the magnitude of their resultant.

CHAPTER XVII.

THE INCLINED PLANE.

186. Equilibrium on a Smooth Inclined Plane.—The conditions of equilibrium of a weight resting on an inclined plane may be found very readily by means of either the Triangle of Forces or the Principle of Work, when the weight is either pushed against the plane by a horizontal force, or is supported by a force acting along the plane.

The force employed to support or raise the weight is sometimes called the *effort or power*.

The *length*, *base*, and *height* of the plane will sometimes be denoted by the letters l, b, h; for the meaning of these terms see § 152.

In diagrams it is usual to represent an inclined plane by its section ABC.

For experimental treatment see § 333.

187. Equilibrium on an Inclined Plane under a Supporting Force applied *horizontally*.

Let a body of weight W be supported at any point O on the plane ABC by a horizontal force P.

It is required to find P, the dimensions of the section ABC being supposed given.

The three forces which keep the body in equilibrium are :

(i) The weight W acting vertically downwards, and therefore perpendicular to AB.

(ii) The applied force P acting horizontally, and therefore perpendicular to BC.

(iii) The reaction of the plane, R, which, since the plane is smooth, acts perpendicular to CA.

Therefore the three forces W, P, R act perpendicular to AB, BC, CA respectively.

Turn the inclined plane round, through a right angle, into the position **DEF**, so that its base **DE** is now vertical and its height **EF** horizontal (Fig. 69). Then the forces W, P, R are parallel to **DE**, **EF**, **FD**.

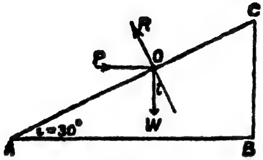

Fig. 68.

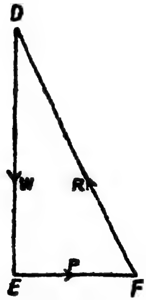

Fig. 69.

Hence, by the Triangle of Forces, the three sides of the triangle **DEF** can represent the three acting forces W, P, R both in direction and in magnitude.

Also the triangles **DEF**, **ABC** are equal in all respects;

therefore $$\frac{P}{\mathbf{EF}} = \frac{W}{\mathbf{DE}} = \frac{R}{\mathbf{FD}}, \text{ or } \frac{P}{\mathbf{BC}} = \frac{W}{\mathbf{AB}} = \frac{R}{\mathbf{CA}}.$$

Therefore $$\frac{P}{h} = \frac{W}{b} = \frac{R}{l},$$

or $$P = W \times \frac{\text{height of plane}}{\text{base of plane}} \quad \text{.................(1)},$$

$$R = W \times \frac{\text{length of plane}}{\text{base of plane}} \quad \text{.................(2)}.$$

Since action and reaction are equal and opposite (by Newton's Third Law), the weight presses against the plane with a force equal and opposite to the reaction of the plane, and the magnitude of this force of pressure is therefore R and is given by (2).

Corresponding to the condition that three forces in equilibrium must be representable by the sides of a triangle *taken in order*, we have the obvious condition that, for equilibrium, P must be a *pushing* force, as

represented by the arrow, and not a *pull* in the opposite direction. Generally, this condition may be thus expressed : If three forces at a point are in equilibrium, and lines be drawn from the point in the sense (§ 165) of the forces, each line must lie outside the geometrical angle formed by the other two. *To illustrate this rule*, the line of action of P should be drawn on the right of O, with the arrow pointing from O.

Example.—To find the horizontal force which will support a weight of half a ton on an incline of 30°.

Here BAC is a semi-equilateral triangle ;

$$\therefore \quad BC = AB/\sqrt{3} = AB \times \tfrac{1}{3}\sqrt{3}, \quad \text{or} \quad h = b \times \tfrac{1}{3}\sqrt{3} ;$$

$$\therefore \quad P = W \times \tfrac{1}{3}\sqrt{3} = 10 \text{ cwt.} \times \tfrac{1}{3}\sqrt{3},$$

required force = 5·7735 ... cwt.

= 5 cwt. 3 qr. 2¾ lb. wt. nearly.

NOTE. *The following results should be verified by the student as an exercise* :—

If the inclination of the plane is

$$0°, \qquad 30°, \qquad 45°, \qquad 60°,$$

the horizontal force required to support W is

$$0, \qquad \sqrt{\tfrac{1}{3}}W, \qquad W, \qquad \sqrt{3}W,$$

and the force of pressure on the plane is

$$W, \qquad \sqrt{\tfrac{4}{3}}W, \qquad \sqrt{2}W, \qquad 2W.$$

188. Equilibrium on an Inclined Plane, the Supporting Force acting *along the Plane*.

Let a given weight W rest on a smooth inclined plane of given section ABC, and let it be kept from sliding down by a force P acting up the plane.

It is required to find the magnitude of P.

Let R be the reaction of the plane.

Then the forces acting on the weight are

(i) W, acting vertically downwards ;

(ii) P, acting in the direction AC ;

(iii) R, acting perpendicular to the plane (since the plane is smooth).

Produce the vertical CB to D, and make CD = CA. Also draw DE perpendicular on the plane. Then the triangles ABC, DEC are equal in all respects, and therefore

$$EC = BC, \quad DE = AB.$$

Now, the forces P, W, R are parallel to the sides **EC, CD, DE** of the triangle **DEC**. Therefore, by the Triangle of Forces,

$$\frac{P}{\mathrm{EC}} = \frac{W}{\mathrm{CD}} = \frac{R}{\mathrm{DE}}; \text{ or } \frac{P}{\mathrm{BC}} = \frac{W}{\mathrm{CA}} = \frac{R}{\mathrm{AB}},$$

whence
$$\frac{P}{h} = \frac{W}{l} = \frac{R}{b}.$$

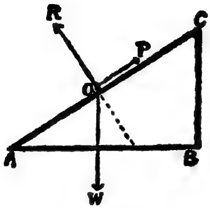

Fig. 70.

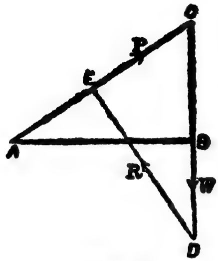

Fig. 71.

Therefore
$$P = W \times \frac{\text{height of plane}}{\text{length of plane}} \quad\text{..................}\quad (3),$$
$$R = W \times \frac{\text{base of plane}}{\text{length of plane}} \quad\text{..................}\quad (4).$$

Example.—To find the force acting up an incline of 30° that will support a weight of $\frac{1}{2}$ cwt.

In this case the required force

$$= \text{weight} \times \frac{\text{height of plane}}{\text{length of plane}} = \tfrac{1}{2}\text{ cwt.} \times \tfrac{1}{2}$$
$$= \tfrac{1}{4}\text{ cwt.} = 28\text{ lb. wt.}$$

NOTE. *The following results should be verified by the student as an exercise :—*

If the inclination of the plane is

$$0, \qquad 30°, \qquad 45°, \qquad 60°, \qquad 90°,$$

the force up the plane which will support W is

$$0, \quad \sqrt{\tfrac{1}{4}}W, \quad \sqrt{\tfrac{2}{4}}W, \quad \sqrt{\tfrac{3}{4}}W, \quad \sqrt{\tfrac{4}{4}}W,$$

and the force of pressure on the plane is

$$\sqrt{\tfrac{4}{4}}W, \quad \sqrt{\tfrac{3}{4}}W, \quad \sqrt{\tfrac{2}{4}}W, \quad \sqrt{\tfrac{1}{4}}W, \quad 0.$$

189. Alternative Method.—*The expression for P also follows very simply from the Principle of Work.*

Let the force P applied along an inclined plane pull the weight W from the bottom to the top of the plane with uniform velocity.* Then P moves its point of application along the length AC, and the weight W is raised against gravity through the vertical height BC of the plane. Equating the two amounts of work, we have

$$P \times \text{length of plane} = W \times \text{height of plane},$$

$$P = W \times \frac{\text{height of plane}}{\text{length of plane}} \quad \text{......................(3).}$$

[When one smooth body slides on another, no *work* is done by or against their reactions.]

Example.—A road rises 440 feet in a mile. To find the pull that a horse must exert on a cart weighing 6 cwt. to draw it up the road.

Let the force be P cwt. Then work done by P in moving its point of application through 1 mile = work required to lift 6 cwt. through 440 feet; $\therefore \quad P \times 5280 = 440 \times 6$;

$$\therefore \quad P = \frac{6 \times 440}{5280} = \frac{6}{12} = \tfrac{1}{2} \text{ cwt.} = 56 \text{ lb. wt.}$$

190. Equilibrium on an Inclined Plane, the Supporting Force being applied in any direction whatever.

When the supporting force P is applied in any direction other than those considered above, its magnitude can, in general, only be calculated by Trigonometry, but it may be determined graphically thus:—

On the vertical through O measure OD downwards containing as many units of length as there are units of force in the weight W. Draw DF perpendicular to the inclined plane, and let it meet the line in which P is applied in the point E. Then, by the Triangle of Forces, EO represents the force P, and DE represents the reaction R. Hence, if the figure is carefully drawn, P and R can be found by measuring the lengths EO, DE.

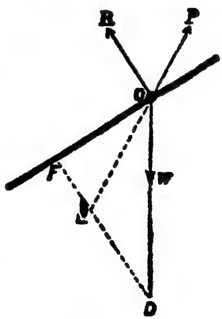

Fig. 72.

For different directions of P, the point E always lies on the straight line DF. Evidently EO is least when E is at F, because the perpendicular OF is less than any other straight line drawn from O to the line DE.

* If the velocity were not uniform, the forces W, P, R would not be in equilibrium, and, moreover, we should have to take account of the work done in producing kinetic energy.

Hence *the force required to support a given weight is least when it acts along the plane.*

To find P and R, having given the inclination **FOE**.

We find from the triangle **ODF** (as in § 188) that

$$OF = W \times h/l, \quad FD = W \times b/l;$$

and, knowing **OF**, the other sides **EO**, **FE** of the right-angled triangle **OFE** can generally be found, and hence **DE** is also known.

191. **The Triangle of Forces** can often be applied to the equilibrium of weights supported by strings, rods, or inclined planes, when it is required to calculate the supporting forces. In drawing a diagram to represent these, it frequently happens that certain lines naturally form a triangle of forces, and the problem is then very simple.

The following may be taken as types of such problems :—

Examples.—(1) In the crane **ACB**, the jib or rod **CA** is 12 ft. long, and is connected to the wall **BC** by a chain **AB**, 8 ft. long attached at a point **B** 6 ft. above **C**.

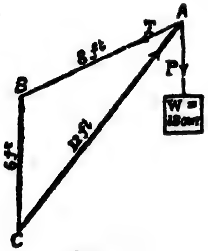

Fig. 73.

To find T the pull of the chain and P the thrust of the rod, when a weight W, equal to 18 cwt., is hung from **A**.

The forces at **A** are

(i) T along **AB**,

(ii) P along **CA**,

(iii) W or 18 cwt. acting vertically, that is, parallel to **BC**.

Hence these forces are parallel to the sides of the triangle **ABC**.

Therefore, by the Triangle of Forces, T, P, W can be represented in magnitude by **AB**, **CA**, **BC**.

But $\quad AB = 8$ ft., $CA = 12$ ft., $BC = 6$ ft.;

$$\therefore \quad AB = \tfrac{4}{3}BC \text{ and } CA = 2BC;$$

$$\therefore \quad T = \tfrac{4}{3}W \text{ and } P = 2W.$$

But $\quad W = 18$ cwt.;

$$\therefore \quad \text{tension of chain } T = \tfrac{4}{3} \times 18 = 24 \text{ cwt.,}$$

and $\quad$ thrust of jib $P = 2 \times 18 = 36$ cwt.

For experimental treatment see § 334.

(2) A ball 1 ft. in radius, and weighing 5 lb., rests against a smooth wall, and is attached to a string which passes through a hole in the wall at **A**, and is pulled with a force of 10 lb. To find the length of string projecting from the hole.

Let R be the reaction of the wall.

The forces on the ball are—

(i) Its weight 5 lb. acting vertically through its centre **C**;

(ii) The reaction R acting perpendicular to the wall, and therefore in direction **BC**;

(iii) The pull of the string, or 10 lb., in direction **CA**.

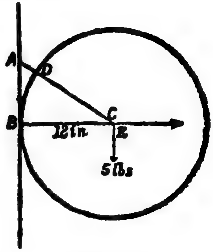

Fig. 74.

The three forces are represented in direction by **AB**, **BC**, **CA**.

Therefore, if the length **AB** represents the weight 5 lb., **BC** will represent the reaction R and **CA** the pull of 10 lb.

Now **CBA** $= 90°$ and **AC** $=$ 2**AB**;

$\therefore\ \angle$**BCA** $= 30°$, $\angle$**BAC** $= 60°$;

$$\therefore\quad BC = AB\sqrt{3};$$

whence $$R = 5\sqrt{3}\text{ lb.} = 8{\cdot}66\ldots\text{lb.}$$

But $$BC = 12 \text{ inches};$$

$$\therefore\quad AC = \frac{2}{\sqrt{3}}\,BC = \frac{24}{\sqrt{3}}\text{inches} = 8\sqrt{3}\text{ inches};$$

and, if **D** is the point of attachment of the string,

$$AD = AC - DC = 8\sqrt{3} - 12 = 1{\cdot}856 \text{ in.}$$

Therefore the required reaction is 8·66 . . . lb., and 1·856 in. of the string projects from the wall.

(3) A weight of 1 ton is attached at **B** to a rod **AB**, which is drawn aside from the vertical position through 30° by a chain **BD** attached to **B**. Find the pull in the rod, supposing **BD** to make an angle of 60° with the downward drawn vertical.

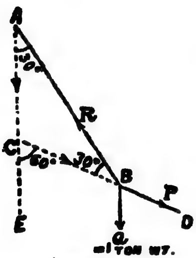

Fig. 75.

Let R be the required pull in the rod **BA**, P the pull in the chain **BD**, Q (= 1 ton) the given weight.

Take any point **A** on the rod, and let the vertical through **A** meet **DB** produced in **C**.

Then the forces P, Q, R are represented in direction by **CB**, **AC**, **BA**. Therefore they can also be represented in magnitude by these lines.

In the figure,

$$\angle BAC = 30°,\quad \angle ECB = 60°;$$

and $$\therefore\quad \angle CBA = 30°.$$

Therefore **ACB** is an isosceles triangle having its base angles each 30°, and if **C** be joined to the middle point of **AB**, the triangle **ACB** will be divided into two triangles whose angles are 30°, 60°, 90°.

$$\therefore\quad AB = 2\times\frac{\sqrt{3}}{2}AC = \sqrt{3}\ AC.$$

$$\therefore\ \text{required pull } R = \sqrt{3}\,Q = \sqrt{3} \text{ tons weight.}$$

EXAMPLES XVII.

1. Find the horizontal forces required to support the following weights on the given inclined planes—

(i) 12 lb. on an incline of length 10 ft. and height 6 ft.;

(ii) 60 lb. on an incline of height 7 ft. and base 24 ft.;

(iii) 180 kilog. on an incline of 11 in 61 of length 122 metres.

2. Find also the works done in drawing each of the weights of Ex. 1 up the plane on which it is placed.

3. Find the forces up the plane required to support the following weights on the given inclined planes—

(i) 117 lb. on an incline of length 13 ft. and height 5 ft.;

(ii) 40 tons on an incline of 7 in 25 of length 50 yd.;

(iii) 170 grammes on an incline of length 85 cm. and base 84 cm.

4. Find the works done in drawing each of the weights of Ex. 3 up the plane on which it is supported.

5. Find the horizontal force, and also the force up the plane, required to support a weight of

(i) 36 oz. on an incline of 30°;

(ii) $12\sqrt{2}$ lb. on an incline of 45°;

(iii) 120 grammes on an incline of 60°.

6. Find in each case of Ex. 5 the reaction of the plane.

7. The lengths of the three inclined planes of Ex. 5 are 24 feet, 5 yards, and 75 centimetres respectively. Find the works done in drawing the weights up the planes.

8. A weight resting on a smooth inclined plane is held in position by a horizontal force. Draw a diagram exhibiting the directions and magnitudes of the forces acting, and calculate the force necessary to hold 1 cwt. on a plane tilted 30° from the level (*a*) if acting horizontally, (*b*) if acting at the best angle.

9. A weight W is placed on an inclined plane, and it is found that it can be supported by either a horizontal force $2P$ or a force P along the plane. Find the inclination of the plane.

10. An inclined plane is 20 ft. long, and a horizontal force of 30 lb. supports on it a weight of 40 lb. Show that, if the plane be still further tilted until its top is 3 ft. higher than before, and the force act along the plane, it will still support the weight.

*11. A weight of 576 lb. is supported on an incline of 16 in 65 by two equal forces, one acting horizontally and the other along the plane. Find the forces.

12. If a force P acting along an inclined plane can support a weight W, and when acting horizontally can support a weight w, show that

$$P^2 = W^2 - w^2.$$

13. Find into what two parts a weight of 50 lb. must be divided so that one part hanging over the top of a plane whose length is 13 ft. and height 5 ft. may balance the other part resting on the plane.

*14. A body, whose mass is 5 kilog., rests on a smooth plane inclined at an angle of 30° to the horizon, and is supported by two forces. One of these forces is equal to the weight of 2 kilog. and acts upwards along the plane, and the other is a force P acting upwards at an angle of 30° to the plane. Determine the value of P when there is equilibrium.

15. A weight of 10 lb. is suspended by two strings, which are inclined to the vertical at angles of 45°. Find the pull in each string.

16. A weight of 16 lb. is supported by two strings, which are inclined at angles of 30° and 60° to the vertical respectively. Find the tension of the strings.

*17. An iron ball weighing 1 lb. is held up by a string, the upper end of which is fixed. If the ball be drawn aside until the string makes an angle of 30° with the vertical and then let go, state clearly what forces act on the ball at the moment of release.

18. A stone, weighing 1 ton, is suspended in the air by a chain; a rope fastened to the stone is pulled so that the chain makes an angle of 30°, and the rope an angle of 60°, with the vertical. Draw a very careful figure, showing the three forces acting on the stone, and a triangle representing them. Find the pull on the rope.

19. A weight of 20 lb., suspended by a string from a peg P, is pulled aside by another string knotted to the first at a point K and pulled horizontally. Find the force necessary to pull it until PK is 60° from the vertical; and find, at the same time, the force on the peg.

20. A picture, weighing 56 lb., is slung over a nail in the ordinary way by a cord attached to two eyes in the top horizontal bar of its frame. If the height of the nail above this bar is half the distance between the eyes, what is the tension in the cord? Under what circumstances would the tension be equal to or greater than the whole weight of the picture?

21. A picture is hung by a string fastened at two corners of its frame and passing over a smooth nail. What are the forces producing equilibrium, and how will the pull on the string vary as it is made longer or shorter?

22. A weight of 12 lb. is suspended by two strings of equal length attached to two pegs in the same horizontal line. If the strings will just break under a tension of 12 lb., find the greatest angle that they can make with each other.

23. A picture, weighing 40 lb., is hung with its upper and lower edges horizontal by a cord fastened at its two upper corners and passing over a nail so that the parts of the cord at the two sides of the nail make an angle of 60° with each other. Find the pull in the cord in pounds weight.

*24. A weight of 6 lb. is supported on an inclined plane by a certain force; the inclinations of the force to the inclined plane and of the inclined plane to the horizon are each 30°. Find the reaction of the plane.

25. A weight of 5 lb. rests on a smooth plane inclined at an angle of 45° to the horizon. Find the least value of the force required to support it, and the thrust on the plane.

*26. A body, weighing 16 lb., is placed on a smooth plane inclined at an angle of 30° to the horizon. Find the two directions in which a force equal to the weight of the body may act to produce equilibrium; also find the reaction of the plane in each case.

27. A mass of 30 lb. is supported by two strings, one of which is horizontal and the other is inclined at an angle of 30° to the vertical. Find the tensions of the strings.

28. AB and AC are two chains 9 ft. and 12 ft. long attached to pegs B, C at a horizontal distance of 15 ft. apart. Find the pulls in the chains when a weight of 1 ton is suspended from A.

CHAPTER XVIII.

TRANSMISSION OF FORCE: THE WEDGE.

192. Rigid Bodies.—In treating the conditions of equilibrium of several forces acting "*at a point*," we have supposed the forces to be all applied to a single particle placed at that point. When two or more forces act *in parallel straight lines*, it is impossible to suppose them to be applied to the same particle, for parallel lines never meet. They must, therefore, be supposed to be applied to a body of extended size. Accordingly, it will be necessary to state what is meant by a rigid body before proceeding further.

DEFINITION.—*A* **rigid body** *is a body whose size and shape always remain the same whatever forces be applied to different parts of it.*

By this it is implied that *the distance between any two particles of a rigid body always remains the same.*

In reality no body is perfectly rigid, but many solid bodies may be regarded as rigid for all practical purposes.

193. Statical Property of a Rigid Body.—In the first place, it will be observed that

Two forces acting at two points of a rigid body are in equilibrium if, and only if, they are equal and opposite, and act in the same straight line.

This may readily be verified by attaching strings to two points **A**, **B** of a body (say a board) resting on a table (Fig. 76), and pulling their ends apart horizontally. The body will turn round until the strings **MA**, **BN** are both in one straight line, and will then come to rest. And, since the forces produce no motion of the body as a whole (*i.e.* no motion of translation), they must be equal and opposite.

[If the body is again displaced so as to bring the strings out of one straight line, it will not remain at rest, but will rotate back to its former position. Hence two equal and opposite forces which do *not* act in the same straight line are not in equilibrium, but tend to produce rotation.]

From the above property we deduce the following principle :—

194. Principle of Transmission (or Transmissibility) of Force—

A force acting on a rigid body may have its point of application transferred to any point whatever in the straight line in which it acts without affecting the conditions of equilibrium.

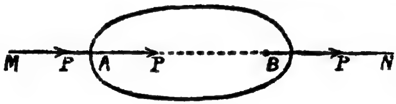

Fig. 76.

Let P be any force applied to a body at **B** in the direction **BN**. Let **A** be any point of the body in the straight line **BN** or **BN** produced. At **A** apply two equal and opposite forces of magnitude P in the straight line **AB**. These two forces balance each other, and therefore do not affect the conditions of equilibrium of the original forces. Now consider the two forces $+P$ at **B** and $-P$ at **A**. By the property just proved, these two forces balance each other, and therefore they can be removed without affecting the conditions of equilibrium. We are, therefore, left with the force $+P$ at **A** as the statical equivalent of the original force P at **B** applied in the same straight line ; as was to be proved.

The principle may also be stated thus—

When a force acts on a rigid body, it is immaterial what point in its line of action is considered to be the point of application of the force.

The point of application may even be taken *outside the body*, provided that the force is applied to a particle rigidly connected with the body. But a force *cannot* be replaced by an equal and parallel force acting at any point *not* in its original line of action.

Tension of a String.—In the same way it is evident that *the tension of a stretched string is the same at all points of its length.* This property is very important.

195. Conditions of Equilibrium of Three Forces in One Plane.

If a rigid body is in equilibrium under three forces in one plane, their lines of action must all be parallel or all pass through one common point.

For let the three forces be not all parallel. Then the lines of action of two of them must meet in some point, say **O**. By the principle of Transmission of Force, we may suppose these two forces to be applied to a particle of the body (or rigidly connected with the body) at **O**. Hence, as in Chap. XV., they are equivalent to a single resultant force acting at **O**. This resultant and the third force balance; therefore they must be equal and opposite and in the same straight line (§ 193). Hence the line of action of the third force must pass through **O**, and therefore the three lines of action must all pass through one common point.

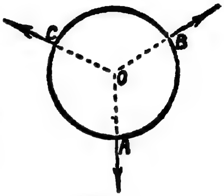

Fig. 77.

In addition to passing through one common point, the forces must be capable of being represented in magnitude and direction by the sides of a triangle taken in order.

The conditions of equilibrium of three *parallel* forces will be investigated in Chap. XX.

196. *The point of intersection of the forces need not be within the body.*

Thus, let three cords be attached to a ring or hoop at the equidistant points **A**, **B**, **C** (Fig. 77), and let these cords be pulled with equal forces in the direction of radii **OA**, **OB**, **OC**. Then these forces will be in equilibrium, for their directions pass through one common point (viz. the centre **O**), and are inclined at angles of 120°. Hence the ring will remain at rest notwithstanding the fact that the point **O** is not in the substance of the ring itself. (See also § 199, Ex. I.)

197. **The wedge** is a triangular block which is used either for splitting a body (*e.g.* a piece of wood) into two parts, for separating two bodies, or for slightly raising heavy weights off the ground. The section of the block is a triangle, and the wedge generally studied on account of its greater utility and simplicity is isosceles, and could be formed by two right-angled inclined planes put back to back. A knife and a chisel afford good illustrations of the principle of the wedge.

198. **Conditions of Equilibrium of a Smooth Wedge.**—Let a smooth wedge, whose section is the triangle **ABC**, be driven in between two bodies **M**, **N** by a force P applied on its face **BC**. Let Q, R be the reactions with which **M**, **N** resist the insertion of the wedge.

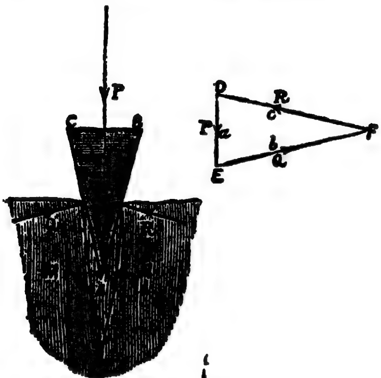

Fig. 78.

Then the wedge is kept in equilibrium by the three forces P, Q, R, and, since the wedge is smooth, these forces act perpendicularly to **BC**, **CA**, **AB**, their directions meeting in some point **O**.

Therefore, by the Perpendicular Triangle of Forces (or as shown by the triangle **DEF**), P, Q, R can be represented in magnitude by **BC**, **CA**, **AB**, respectively, that is,

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}};$$

or, if a, b, c denote the lengths of **BC**, **CA**, **AB**,

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \qquad \text{.........................(1).}$$

In the case generally considered the section of a wedge is isosceles, or $b = c$. Therefore

$$Q = R = P \times \frac{c}{a}.$$

By sufficiently increasing the length b and making the breadth a very small, a very small force P can be made to overcome a very large resistance Q.

199. Equilibrium of a Heavy Body: Applications to Problems.—The theorem of § 195 is very useful in enabling us to find the conditions of equilibrium of a heavy body supported at two given points by forces that are not vertical. The cases where the supporting forces are vertical will be considered later.

It will be proved in Chap. XXIV. that the weight of a rigid body may always be supposed to act vertically at a single point of the body, called its *centre of gravity* or *centre of mass.* For the present, the following particular results will be assumed :—

(1) The weight of a heavy *uniform rod* or *beam* acts at its middle point.

(2) The weight of a *uniform sphere* or *cube*, or of a *circular disc*, acts at its centre.

It will also be necessary to remember that—

The reaction of a perfectly smooth surface is always perpendicular to that surface.

Examples.—(1) *Equilibrium of a ladder.*—A uniform ladder of weight W leans against a perfectly smooth wall. To find the thrusts which it exerts against the wall and ground when the ladder is 20 ft. long and reaches a height of 16 ft.

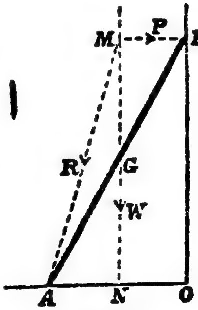

Fig. 79.

Let AB be the ladder, G its middle point.

Let P denote the reaction of the wall, R that of the ground.

Then the three forces acting on the ladder are—

(i) Its weight W acting vertically through G (since the ladder is uniform);

(ii) The reaction P at B acting horizontally (since the wall is smooth and vertical);

(iii) The reaction R acting at A.

Since these forces are in equilibrium, they must pass through one point. Let the vertical through G meet the horizontal through B in M. Then the reaction R must act in the line AM.

Let MG meet the ground in N. Then the forces W, P, R act in the directions of MN, NA, AM, the sides of the triangle AMN taken in order. Therefore, by the Triangle of Forces, these sides may be made to represent the forces in magnitude, so that if BO or MN represents W, NA and AM will represent P and R respectively. Therefore

$$P = W \times \frac{\mathrm{NA}}{\mathrm{MN}}, \quad R = W \times \frac{\mathrm{AM}}{\mathrm{MN}}.$$

Now $$\mathrm{AB} = 20 \text{ ft.}, \quad \mathrm{BO} = 16 \text{ ft.};$$

$$\therefore \quad \mathrm{AO}^2 = 20^2 - 16^2 = 4^2(5^2 - 4^2) = 4^2.3^2 = 12^2;$$

$$\therefore \quad \mathrm{AO} = 12 \text{ ft.};$$

and it is easily seen* that

$$\mathrm{AN} = \tfrac{1}{2}\mathrm{AO} = 6 \text{ ft.}$$

Therefore also

$$\mathrm{AM}^2 = \mathrm{AN}^2 + \mathrm{NM}^2 = 6^2 + 16^2 = 2^2(3^2 + 8^2) = 2^2 \times 73;$$

$$\therefore \quad \mathrm{AM} = 2\sqrt{73} \text{ ft.}$$

$$\therefore \quad \text{Reaction of wall } P = W \times \frac{6}{16} = \frac{3W}{8},$$

$$\text{Reaction of ground } R = W \times \frac{2\sqrt{73}}{16} = \frac{W\sqrt{73}}{8}.$$

Indicate the forces which maintain (i) a uniform door, (ii) a kite, in equilibrium.

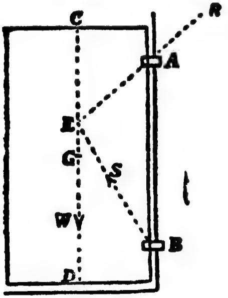

Fig. 80.

(i) For the door, let CD be the median line; then G, its middle point, will be the centre of mass of the door, *i.e.* the point of action of its weight. In brief, the weight acts along CD. The reactions R and S at the hinges A and B must act so that their lines of direction meet at some point E in CD. (What precise point this will be is indeterminate without further data.) The forces S, R, W will then be represented by BE, EA, AB.

NOTE.—In a *gate* which merely rests against a peg at B, the reaction BE is horizontal, and the forces S, R are completely determined.

(ii) For the kite, we must premise that, owing to the absence of friction, whatever be the *direction* of the wind, the only effect of it will be to produce a pressure perpendicular to the face of the kite, and the force due to this pressure may be taken to act at the centre of gravity of the face, say at C, on the stem AB. The actual

* For AG = GB; ∴ triangles AGN, BGM are equal in all respects; ∴ AN = BM = NO (the opposite side of the parallelogram MBON); ∴ AN = ½AO.

centre of gravity of the kite, with its tail, will be at **G**, somewhat lower down on **AB**; thus W (the kite's weight) will act vertically through **G**. Let the directions of F and W meet at a point **D** behind the kite. Then, at any moment when the kite is stationary, the direction of the tension T of the string that holds the kite must pass through **D**. Moreover, it will be seen from the figure that the line of action of T must be *above* that of F. Hence the point of attachment of the string

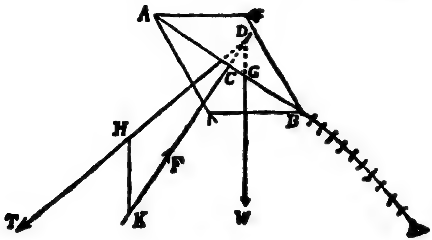

Fig. 81.

should be slightly above the centre of the face of the kite. Take any convenient point **H** on the string, and draw **HK** vertically to meet the direction of F at **K**. Then the triangle **HDK** is a triangle of forces, and shows us that the weight of the kite and the tension in the string must be smaller than the force of pressure of the wind. (For, since **KHD** is obtuse, **KD** is the greatest side of the triangle.)

(3) A uniform rod **AB** weighing 1 cwt., hinged at **A**, is supported in a horizontal position by a rope attached to **B**, and making an angle of 45° with the rod. To find the tension in the rope and the force at the hinge.

Let **G** be the middle point of **AB**.

Let P be the tension in the rope, Q the force at the hinge, and let W denote the weight of the beam (1 cwt.).

Then the forces acting on the rod are—

(i) Its weight 1 cwt. acting vertically through G.

(ii) The tension P acting along the string at **B**.

(iii) The reaction Q of the hinge at **A**.

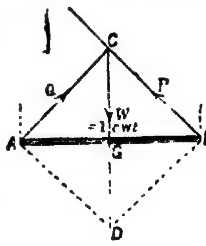

Fig. 82.

These three forces must all pass through one point. Let the vertical through **G** meet the rope at **C**. Then Q must act along **AC**. Draw **AD** parallel to **CB**, meeting **GC** produced in **D**.

Then P, Q, W are representable in direction, and therefore also in magnitude, by **DA**, **AC**, **CD**.

Since $\angle ABC = 45°$ and $GA = GB$, every triangle in the figure is easily seen to be a right-angled isosceles triangle, and every angle in the figure is either 90° or 45°. In the triangle DAC,

$$DA = AC = \tfrac{1}{2}\sqrt{2}.CD;$$

$$\therefore \quad P = \quad Q = \tfrac{1}{2}\sqrt{2}W = \tfrac{1}{2}\sqrt{2}\text{ cwt.}$$

Therefore the tension in the string is $\frac{1}{2}\sqrt{2}$ of a cwt., and the force at the hinge is also $\frac{1}{2}\sqrt{2}$ of a cwt., and its direction makes an angle 45° with the horizon.

EXAMPLES XVIII.

1. A heavy uniform ladder rests with its upper end against a smooth vertical wall; show by a figure how to determine the direction of the resultant force acting upon the foot of the ladder.

2. A uniform straight rod is supported by means of two strings attached to a fixed point and to the ends of the rod. Show that the tensions of the strings are proportional to their lengths.

*3. A uniform heavy rod, whose length is 8 ft., is placed across a smooth horizontal rail, and rests with one end against a smooth vertical wall, the distance of which from the rail is $\frac{1}{2}$ ft. Show that it will be in equilibrium if the angle which the rod makes with the vertical is 30°.

4. A uniform heavy rod, whose weight is 16 lb., is movable about its lower extremity and rests against a smooth vertical wall. The rod makes an angle of 45° with the wall; find the thrust of the rod against the wall.

5. A weightless rod, 3 ft. long, is supported horizontally, one end being hinged to a vertical wall, and the other attached by a string to a point 4 ft. above the hinge; a weight of 180 lb. is hung from the end supported by the string. Calculate the tension of the string and the pressure along the rod.

6. A rod AB is hinged at A and is supported in a horizontal position by a string BC, making an angle of 45° with the rod. The rod has a weight of 10 lb. suspended from B. Find the tension in the string and the force at the hinge. The weight of the rod may be neglected.

*7. A uniform rod, whose length is 12 ft. and weight 20 lb., is placed over a smooth peg so that one end rests against a smooth vertical wall. The distance of the peg from the wall is 9 inches. Find the position of equilibrium and the force of pressure on the peg.

*8. A uniform rod, whose length is 12 ft. and weight 12 oz., is placed over a smooth peg, so that one end rests against a smooth vertical wall. The thrust on the peg is 24 oz. Find the distance of the peg from the wall and the position of equilibrium.

*9. A pole, 24 ft. long, weighing 60 lb., rests with one end against the foot of a wall, and from a point [illegible] ft. from the other end a cord runs

horizontally to a point in the wall 16 ft. from the ground. Find the tension of the cord, and the pressure of the lower end of the pole on the ground.

10. A rectangular box, containing a ball of weight 12 lb., is placed on an incline of 30° to the horizon. The box is prevented from moving on the plane by a small projection. Find the thrusts between the ball and the box.

11. A uniform rod, 10 ft. long and weighing 25 lb., is hinged at one end, and to the other end is fastened a rope 16 ft. long, which is also fastened to a point 10 ft. vertically above the hinge. Find the tension of the rope and the reaction of the hinge.

12. A rod AB, weighing 12 lb., is hinged at A and supported in a horizontal position by a string BC making an angle of 45° with the rod. Find the tension of the string.

*13. A uniform rod AB, inclined at an angle of 30° to the horizon, and weighing 8 lb., rests with the end A against a rough horizontal table, the end B being supported by a string attached to a point C vertically above A. If BC be at right angles to AB, find the reaction of the table and the tension of the string.

14. A uniform rod, whose length is 8 in. and weight 2 lb., is supported with its upper end resting against a rough vertical wall by means of a string attached to the lower end of the rod and to a point in the wall 8 inches vertically above the end of the rod against the wall. Find the tension of the string and the pressure against the wall, if the string makes an angle of 30° with the rod.

15. AC and BC are two smooth inclined planes at right angles to one another, and intersecting at their lowest point C. A uniform heavy rod AB rests in equilibrium against them. Show that its middle point is vertically above C.

16. Construct a triangle whose sides represent the forces acting on the rod in Ex. 15, and calculate the forces of pressure of the rod against the planes, the inclinations of the planes to the horizon being 30° and 60°, and the weight of the rod 1 lb.

*17. A pole 12 ft. long, weighing 25 lb., rests with one end against the foot of a wall, and from a point in the wall 6 ft. vertically above the bottom of the pole, a cord runs horizontally to a point in the pole 2 ft. from its top extremity. Find the tension of the cord, and the pressure of the lower end of the pole

18. A uniform rod, 15 in. long and weighing 12 lb., is suspended by two strings, 9 and 12 in. long, respectively, which are attached to the ends of the rod and also to a nail above the rod. Find the position of equilibrium and the tension of each string.

19. A wedge, whose triangular section is isosceles, and has its sides 10, 13, and 13 in. long, is used in splitting a stone. If the force applied to the back of the wedge be 200 lb., find the force of pressure of the stone on each surface of the wedge in contact with it.

EXAMINATION PAPER X.

1. Find the ratio of the effort to the weight on an inclined plane when the effort acts parallel to the plane.

2. Find the force up the plane required to support a weight of 125 lb. on an incline whose height is 7 ft. and base 24 ft.

3. Two forces, one of which acts up the plane and the other is horizontal, support a weight of 32 lb. on an incline of 3 in 5. If the horizontal force be 9 lb., find the force acting up the plane. Find also the total thrust on the plane.

4. State and prove the principle of Transmission of Force.

5. Show that, if three forces in one plane keep a body in equilibrium, their lines of action must either meet in a point or be all parallel.

6. A string 14 ft. long has its extremities fastened to two points in a horizontal line which are 10 ft. apart, and a weight of 75 lb. is attached to a point of the string 6 ft. from one end. Find the tensions in the two portions of the string.

7. A picture weighing 12 lb. is supported by a cord which passes over a nail, and the two portions of the string are at right angles to each other, the top of the picture being horizontal. Find the tension of the string and the reaction of the nail.

8. A smooth wedge, whose triangular section is isosceles and has its sides 14, 25, and 25 in. long, is used for separating two stones lying close together. If the force applied to the back of the wedge be 70 lb., find the force of pressure of each stone against the wedge.

9. A plummet, whose weight is W, is immersed in a current of water, the string being held in the hand, and the string ultimately settles in a position in which it makes an angle of 30° with the vertical. Draw a diagram showing the nature of the forces acting on the plummet, and determine the tension of the string.

10. A ladder weighing 60 lb. rests against a smooth wall of a house at an inclination of 60° to the horizon. Find, to two places of decimals, the force exerted by the ground on the ladder.

PART II.

MOMENTS AND PARALLEL FORCES.

CHAPTER XIX.

MOMENTS OF FORCES IN ONE PLANE.

200. Forces Tending to Produce Rotation.—If a body is attached to a fixed axis or hinge about which it can turn freely, we can set the body in motion by applying to it a force in any direction not passing through the axis. And it is easy to verify by a few simple experiments that, the further off from the axis the force is applied, the more effect it has in turning the body.

Consider, for example, a door which can turn about its hinges. To open or shut the door, we apply a force to its handle in a direction perpendicular to the plane of the door, and at a considerable distance from the hinge. If we press against the woodwork of the door at a point very near the hinge, we shall have to exert a much greater effort to set the door turning, while if we lean against the edge of the door and push it directly towards the hinge, it will not turn at all.

In this chapter we shall consider the equilibrium of bodies under forces which tend to turn them about a fixed point or axis; but shall not consider the actual motion of such bodies when the forces cause them to rotate.

201. Definition.—*The* **moment of a force about a fixed point** *is the product of the measure of the force into the perpendicular distance of the point from its line of action.*

Thus, if P be the force, and if **OM** is drawn from any point **O** perpendicular to the line of action of P, the product $P \times$ **OM** is called the **moment** of the force P about **O**. (See § 320.)

The length **OM** may be called the **arm** of the moment. Therefore

$$\textbf{moment} = \textbf{force} \times \textbf{arm}.$$

The product $P \times OM$ becomes zero if either of its factors diminish to zero, that is,

$$\text{if } P = 0, \text{ or if } OM = 0.$$

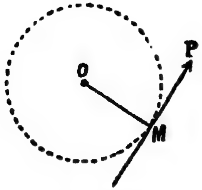

Fig. 83.

In the latter case O is on the line of action of P.

Hence the moment of a force about a point vanishes when either—

(i) the force itself is zero;

(ii) the line of action of the force passes through the point.

If the body be fixed so that it can only turn about O, the force P cannot set the body in motion if its line of action passes through O. (For, by the Principle of Transmission of Force, P may be supposed to act *at* O, and O is prevented from moving by the hinge or support.)

Hence, *when the moment of a force about a fixed point vanishes, the force has no tendency to turn the body about that point.*

On the other hand, if the moment is not zero, the force P cannot pass through O. But the reaction of the hinge or support passes through O. Hence these two forces cannot act in the same straight line, and the body cannot remain in equilibrium. It must therefore turn about O, since no other motion is possible.

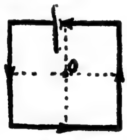

Fig. 84.
Moments positive about O.

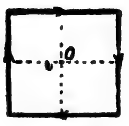

Fig. 85.
Moments negative about O.

202. Positive and Negative Moments.

In forming the algebraic sum of the moments of any number of forces, each moment is taken with its proper algebraic sign, defined as follows:—

Moments which tend to produce rotation in the *opposite* direction to that in which the hands of a watch go are considered positive (Figs. 83, 84).

Moments which tend to produce rotation in the *same* direction as the hands of a watch go are therefore to be regarded as **negative**, and a minus sign is prefixed to their amount (Fig. 85).

Thus, if two forces have equal moments about a point, but tend to produce rotation in opposite directions, the algebraic sum is zero, for one moment is a *plus* and the other a *minus* quantity.

203. Geometrical Representation of the Moment of a Force. *If a force be completely represented by a straight line, its moment about any point shall be measured by twice the area of the triangle which the straight line subtends at that point.*

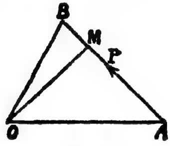

Fig. 86.

Let **AB** represent any force P, then shall the moment of P about **O** be represented by twice the area **OAB**.

Draw **OM** perpendicular to **AB**. Then

$$\text{area of } \triangle \mathbf{OAB} = \tfrac{1}{2}\,\text{base} \times \text{altitude} = \tfrac{1}{2}\mathbf{AB} \times \mathbf{OM}.$$

Now **AB** represents the force P, therefore **AB** must contain P units of length. Hence

$$\text{moment of } P \text{ about } \mathbf{O} = P \times \mathbf{OM} = \mathbf{AB} \times \mathbf{OM} = 2\triangle\mathbf{OAB}.$$

204. If a force P is applied at any point A, its moment about any point O is equal to the product

$$\mathbf{OA} \times \textbf{resolved part of } P \textbf{ perpendicular to } \mathbf{OA}.$$

For let **AC** represent the force P. Complete the rectangle **ABCD**, whose side **AD** passes through **O**. Then, by the Parallelogram of Forces, **AD**, **AB** represent the components of P along and perpendicular to **OA**.

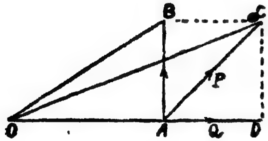

Fig. 87.

Now $\triangle\mathbf{OAB} = \triangle\mathbf{OAC}$

(since they are on the same base and between the same parallels).

Therefore moment of P about **O**

$$= 2\triangle\mathbf{OAC} = 2\triangle\mathbf{OAB}$$
$$= \mathbf{OA} \times \mathbf{AB}$$
$$= \mathbf{OA} \times \text{resolved part of } P \text{ perp. to } \mathbf{OA}.$$

205. The moments of two intersecting forces about any point in the line of action of their resultant are equal and opposite.

Let two forces P, Q act in the lines **AB**, **AD**, and let **C** be any point in the line of action of their resultant R.

Complete the parallelogram **ABCD**.

Choose the scale of representation such that **AD** *represents the force* Q *in magnitude.*

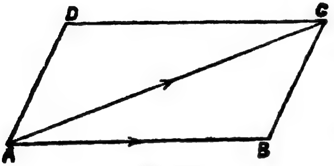

Fig. 88.

Then, by the Parallelogram of Forces, **AB** and **AC** represent the forces P, R.

Since **ABCD** is a parallelogram,

$$\therefore \quad \triangle \mathbf{CAB} = \triangle \mathbf{CDA}.$$

But the triangles **CAB**, **CAD** represent half the moments of P and Q about **C**, and these moments tend to turn about **C** in opposite directions.

Therefore the moments of P, Q about **C** are equal and opposite.

206. The algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about that point (Varignon's Theorem).

Let two forces P, Q act along the lines **AB**, **AD**.

Let **O** be any point in the plane of the forces about which moments are to be taken.

Draw **OD** parallel to **AB** to meet **AD** in **D**.

Choose the scale of representation so that **AD** *represents* Q *in magnitude* (§ 166; also compare § 205), *and on the same scale let* **AB** *be the length which represents* P.*

Let the parallelogram **ABCD** be completed, and join **AC**, so that **AC** represents the resultant R of P, Q.

Case i.—If **O** lies *without* the angle **BAD**, as in Fig. 89, then the moments of P, Q, R about **O** are positive and measured

* This step of the proof should be carefully noted, as it is most important.

by twice the triangles **OAB, OAD, OAC** respectively; hence we have to show that

$$2\triangle \mathbf{OAB} + 2\triangle \mathbf{OAD} = 2\triangle \mathbf{OAC}.$$

Now $$\triangle \mathbf{OAB} = \triangle \mathbf{DAC}$$

($\because$ bases **AB, DC** are equal and **DC** is parallel to **AB**);

$\therefore\quad 2\triangle \mathbf{OAB} + 2\triangle \mathbf{OAD} = 2\triangle \mathbf{DAC} + 2\triangle \mathbf{OAD} = 2\triangle \mathbf{OAC},$

or moment of P + moment of Q = moment of force R.

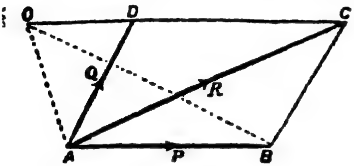

Fig. 89.

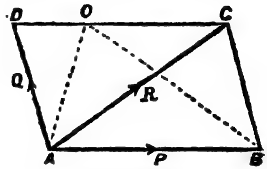

Fig. 90.

Case ii.—If **O** lies *within* the angle **BAD**, say between **AC** and **AD**, as in Fig. 90, the moment of Q is negative and is represented by *minus* twice the area **ODA**, and we have to show that

$$2\triangle \mathbf{OAB} - 2\triangle \mathbf{ODA} = 2\triangle \mathbf{OAC}.$$

Now, as before, $$\triangle \mathbf{OAB} = \triangle \mathbf{DAC},$$

$\therefore\quad 2\triangle \mathbf{OAB} - 2\triangle \mathbf{ODA} = 2\triangle \mathbf{DAC} - 2\triangle \mathbf{ODA} = 2\triangle \mathbf{OAC}.$

Therefore, making allowance for the positive and negative signs of the moments themselves, we have algebraically

moment of P + moment of Q = moment of force R,

as before.

Alternative Proof.—By § 204 (using Fig. 62, p. 171),

Algebraic sum of moments of P, Q about **O**

= **OA**$\times$algebraic sum of resolved parts of P, Q perpendicular to **OA**

= **OA**$\times$resolved part of R perpendicular to **OA** (§ 182)

= moment of R about **O**.

In the next chapter, we shall show that Varignon's Theorem holds good for two parallel forces as well as for two intersecting ones, and, generally, that the algebraic sum of the moments about a point of *any number* of forces in one plane through that point is equal to the moment of their resultant.

207. Equilibrium about a Fixed Point.—If a rigid body, moveable about a fixed point, is kept in equilibrium by two forces in any plane containing that point, the moments of these forces about the point will be equal and opposite.

For, if the forces balance, their resultant must pass through the fixed point. Hence the moments of the forces about that point must be equal and opposite (§ 205).

Either of the forces, if it were to act alone, would set the body turning round about the fixed point. Hence, since the body does not turn when both act, we are led to infer that the tendencies of the forces to produce rotation are equal and opposite.

Hence equal moments about a point represent equal tendencies to produce rotation about that point. Moreover, if a force be doubled, its moment about any point is also doubled; and it is natural to suppose that its tendency to produce rotation about any point is doubled. Hence we infer that the moment of a force about any point is a measure of its tendency to produce rotation about that point.

Example.—A uniform rod AB, of weight W, is hinged at its end A, and held at an inclination of 45° to the horizon by means of a string attached at B, and making an angle of 30° with the rod. To find T, the tension of the string.

The student should draw the figure. Let G be the middle point of AB; then W acts vertically through G. Drop AM perpendicular on its line of action, and AL perpendicular on the string. Equating the moments of T, W, we have

$$T \times \mathrm{AL} = W \times \mathrm{AM}.$$

Now GAM is a right-angled isosceles triangle, and BAL is a semi-equilateral triangle;

$$\therefore\quad \mathrm{AM} = \mathrm{AG} \times \tfrac{1}{2}\sqrt{2} = \mathrm{AB} \times \tfrac{1}{4}\sqrt{2} \quad \text{and} \quad \mathrm{AL} = \tfrac{1}{2}\mathrm{AB},$$

whence we readily find $T = \frac{1}{2}\sqrt{2}.W$.

208. **Conditions of Equilibrium of Coplanar Forces.**—Although it is impossible here to fully investigate the conditions of equilibrium of forces acting *on a rigid body* (that is to say, not all at one point) *in one plane*, yet it is well to note the main results.

To the two conditions of equilibrium given in § 185, must be added a third.

Hence the conditions of equilibrium, when any number of forces in one plane act upon a rigid body, are as follows :—

(*a*) The sums of the resolved parts of the forces along two directions in the plane must be severally zero.

(*b*) The algebraic sum of the moments of the forces about a point in their plane must be zero.

[The former conditions prevent any motion of *translation*, the latter prevents any motion of *rotation*.

The conditions for parallel forces (see Chap. XX.) will be found to be included in these.]

EXAMPLES XIX.

1. Find the moments about a point O of the following forces acting at the extremities of the given arms :—

(i) 4 lb. ; arm, 5 ft. ; (iii) 4 cwt. ; arm, 2 yd. ;
(ii) 16 lb. ; arm, 9 in. ; (iv) 10 grm. ; arm, 5 cm.

2. Find the moment about a point O of a force of 18 lb. acting at a point A along a line AB, where the length of OA is 20 in., and the angle OAB is (i) 30°, (ii) 60°, (iii) 90°, (iv) 135°, (v) 150°.

3. A horizontal rod, 10 ft. long, has a weight of 2 lb. at one end, an upward force of 12 lb. at a distance of 2 ft. from that end, a downward force of 4 lb. at a distance of 6 ft. from the same end, and a weight of 6 lb. at the other end. Taking the first end of the rod as the left-hand end of the figure, write down the moments of the forces about each end of the rod and about its middle point, prefixing the proper sign to each. Also find the algebraic sum of the moments about these points.

4. ABC is an equilateral triangle whose side is 5 in. long. Forces of 2, 4, and 8 lb. act along AB, BC, CA respectively. Find the moment of each force about the opposite angular point.

5. ABC is an isosceles triangle, right-angled at C, and forces of $3\sqrt{2}$, 2, and 4 units act along AB, BC, CA respectively. Find the moment of each force about the opposite angular point.

6. OA, OB are chords, 5 and 12 in. in length, of a circular disc OACB, whose diameter OC is 13 in. If forces of 2 and 5 lb. act along these chords respectively, find how the disc will begin to move if the chords lie on opposite sides of the diameter OC.

7. Two forces, of 5 and 12 lb. respectively, act along the sides AB, AD of the square ABCD. Find the perpendicular distance of the line of action of their resultant from C if the side of the square be 13 in. long.

8. Show that, if two forces be represented in magnitude and direction by two sides of a triangle, taken in order, the sum of their moments about every point in the base is the same.

9. Two forces of 5 lb. and 10 lb. act along the sides AB, AC of an equilateral triangle ABC, whose side is 14 in. Find the perpendicular distances of their resultant from B and C.

10. A system of forces in a plane is such that the sum of their moments about a point O in that plane vanishes. If the forces are not in equilibrium, what do we know about their resultant? (See Note at end of § 206.)

11. Equal forces of 10 lb. each act along the sides **AB**, **BC**, etc., of a regular hexagon **ABCDEF**, taken in order. The side of the hexagon is 4 in. long. Find the moment of each force about the point **A**.

12. Forces of P and Q units act along the sides **AB**, **AD** of a square **ABCD**. Find the perpendicular distances of their resultant from the three corners **B**, **C**, **D**.

13. Forces of 6 and 12 lb. act from **A** to **B** and from **A** to **C** respectively, along the sides of an equilateral triangle **ABC**, whose side is 3 ft. long. Find the moment of each force about the middle point of **BC**.

14. Forces of 2 and 7 lb. act from **A** to **B**, and from **C** to **A**, respectively, along the sides of an equilateral triangle **ABC**, whose side is 10 in. long. Find a point in **BC** about which the algebraic sum of the moments of the two forces will be zero.

15. **ABCD** is a square. Equal forces P act from **D** to **A**, **A** to **B**, and **B** to **C** respectively, and a fourth force $2P$ acts from **C** to **D**. Find the algebraic sum of the moments of these forces about each of the corners in turn.

16. Parallel forces of 2, 4, 6, and 8 lb. respectively, act at intervals of a foot along a straight line, and their lines of action are at right angles to the line. Find a point in the line about which the algebraic sum of their moments is zero.

17. Forces of P and $2P$ units act from **A** to **B** and from **D** to **A** respectively, along the sides of a square **ABCD**. Find the perpendicular distance of the line of action of their resultant from **C**.

18. Forces act along the sides of a triangle, taken in order, and are proportional to the sides along which they act. Show that their moments about the opposite angular points are all equal, and that each moment is represented by twice the area of the triangle.

CHAPTER XX.

PARALLEL FORCES.

209. "Like" and "Unlike" Parallel Forces.—In the preceding chapters we have considered the equilibrium of forces whose directions intersect one another. In the practical applications of mechanics, however, parallel forces are of even more frequent occurrence than intersecting forces.

The following definitions will be required :—

DEFINITION.—Parallel forces which tend in the same direction are said to be **like**. Those which tend in opposite directions are said to be **unlike**.

If a force acting in one direction be regarded as *positive*, it is convenient to regard any unlike force as *negative*.

Thus, if there be forces of 28 lb. acting upwards and 56 lb. acting downwards, the two forces will be unlike, and if we consider the upward direction as positive, the downward one will be negative, and the complete expressions for them will be + 28 lb. and −56 lb. respectively.

In compounding two parallel forces, their resultant, if it exists, may be deduced from the resultant of two intersecting forces by making use of the property that two equal and opposite forces in the same straight line may be applied at any two points of a rigid body without affecting the equilibrium of the body (see §§ 193, 194.)

210. To find the resultant of two *like* parallel forces.

Let P, Q be the two parallel forces, **A**, **B** any two points on their lines of action which we may suppose to be their points of application.

At **A** introduce a force of *any* magnitude F along **AB**; at **B** introduce an equal and opposite force F along **BA** (that is, $-F$ along **AB**); these two forces will balance each other, and will not affect the resultant.

Now compound P with F, and Q with $-F$.

The two former are equivalent to a single force S at **A**, whose direction is *within* the angle **BAP**, and the two latter to a single force T at **B** *within* the angle **ABQ**.

Thus the forces S and T will intersect at a point **M** between the forces P, Q, and will have a resultant R acting at **M**, which will also be the resultant of P and Q.

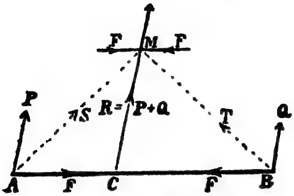

Fig. 91.

Draw **CM** parallel to P or Q, cutting **AB** in **C**.

Replace the force S at **M** by its components along **CM**, and parallel to **AC**. These components are, of course, the same as when S acted at **A**, and are therefore P, F. Similarly, replace the force T at **M** by its components along **CM** and parallel to **BC**; these components are Q, F.

Thus the two forces P, Q or S, T are together equivalent to forces $P+Q$ acting along **CM**, and $F-F$ or zero parallel to **AC**.

Hence the resultant of P, Q is $P+Q$ acting along **CM**; or, in words—

The *magnitude* of the resultant is the sum of the components.

The *direction* of the resultant is parallel to the components and in the same sense.

The *position* of the resultant lies between the two forces, and may be found as follows :—

Since the forces at **A** are parallel to the sides of $\triangle$**ACM**, therefore, by the Triangle of Forces,

$$\frac{P}{F}=\frac{\mathbf{CM}}{\mathbf{AC}}, \quad \text{or} \quad P\times\mathbf{AC}=F\times\mathbf{CM}.$$

Similarly, since the forces at **B** are parallel to the sides of $\triangle$**BCM**,

$$\frac{Q}{F}=\frac{\mathbf{CM}}{\mathbf{CB}}, \quad \text{or} \quad Q\times\mathbf{CB}=F\times\mathbf{CM}.$$

$$\therefore \quad P\times\mathbf{AC}=Q\times\mathbf{CB}.$$

If AB is taken at right angles to the forces, this relation expresses the fact that **the moments of *P*, *Q* about C are equal and opposite.**

We may also write the above relation,

$$\frac{AC}{CB}=\frac{Q}{P};$$

which shows that AC is greater or less than CB, according as Q is greater or less than P. Hence

The resultant is nearer to the greater force.

It divides the line AB into parts which are in the inverse ratio of the forces.

211. When two *unlike* forces P, Q act at A, B respectively, and we apply to the body equal and opposite forces F and $-F$, both acting in the straight line AB, these, compounded with P, Q, give us two new forces S, T, and we must examine whether these intersect or not.

If P, Q be *unlike and unequal* forces, let us suppose P to be the greater, and construct the Parallelograms of Forces ADGH, BEKL, so that AG, BK represent S, T respectively (Fig. 92).

Since $P>Q$, DG>EK. Hence, by placing the triangle DAG with its vertices A, D on B, E (as in proof of Euc. I. 4), we see that ∠DAG> ∠EBK, and it readily follows from Euclid's twelfth axiom that BK produced will meet GA produced through A in a point M on the side of the line of action of P remote from Q (Fig. 93). Therefore S and T or P and Q have a single resultant through M.

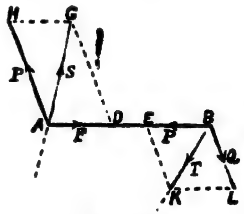

Fig. 92.

212. To find the resultant of two unequal *unlike* parallel forces.

Let P, Q be the forces, P being the greater, A, B any two points on their lines of action. Introduce two equal and opposite forces F, $-F$ at A, B acting along AB. Let S be

the force compounded of P and F, T the force compounded of Q and $-F$. Then the forces S, T lie in the alternate angles at **A** and **B**, and their lines of action will intersect at a point **M** on the side of P remote from Q (§ 211). Draw **MC** parallel to P, cutting **AB** in **C**.

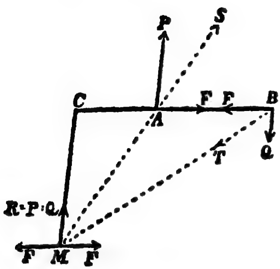

Fig. 93.

Replace the force S at **M** by its components P, F along **MC** and parallel to **AB**, and replace the force T at **M** by its components Q, F along **CM** and parallel to **BA**. Thus the two forces P, Q or S, T are together equivalent to $P - Q$ along **MC**, and $F - F$ or zero parallel to **AB**.

Hence the resultant of P, Q is $\mathbf{P - Q}$ *acting along* **MC**; or in words—

The *magnitude* of the resultant is the difference of the components.

The *direction* of the resultant is parallel to, and in the sense of, the greater component.

The *position* of the resultant lies on the side of the greater force which is remote from the lesser force, and is found thus—

Since the forces at **A** are parallel to the sides of $\triangle$**MCA**,

$$\frac{P}{F} = \frac{\mathbf{MC}}{\mathbf{CA}}, \quad \text{or} \quad P \times \mathbf{CA} = F \times \mathbf{MC}.$$

Similarly, since the forces at **B** are parallel to the sides of $\triangle$**BCM**,

$$\frac{Q}{F} = \frac{\mathbf{MC}}{\mathbf{CB}}, \quad \text{or} \quad Q \times \mathbf{CB} = F \times \mathbf{MC}.$$

$$\therefore \quad P \times \mathbf{CA} = Q \times \mathbf{CB};$$

or the moments of P, Q about C are equal and opposite.

We may also write this relation

$$\frac{\mathbf{CA}}{\mathbf{CB}} = \frac{Q}{P};$$

and, since we have taken Q less than P, CA is less than CB,

which affords another proof that **C** lies on **BA** produced beyond **A.** Hence

The resultant divides the line AB externally into parts which are in the inverse ratio of the forces.

Example.—A man carries a bundle, weight 20 lb., at the end of a stick a yard long over his shoulder, his hand holding the forward end. Find the forces of pressure supported by his hand and his shoulder respectively (i) when 1 foot of the stick's length projects behind his shoulder, (ii) when 2 feet so project.

If P and R denote the forces supported by the hand and shoulder, in the first case, $P \times 2 = 20 \times 1$, or $P = 10$ lb.; hence $R = 20 + 10 = 30$ lb. In the second case, $P \times 1 = 20 \times 2$, or $P = 40$ lb.; hence $R = 20 + 40 = 60$ lb. Thus, by the second arrangement, R is doubled, and P is increased fourfold.

213. **Couples.**—If P, Q are *unlike* and *equal*, the method fails, for the forces S, T are themselves parallel, as is evident from the construction of § 211. Two equal unlike parallel forces are said to constitute a **couple.***

In the present chapter we shall deal with parallel forces that do not form couples. We shall, therefore, always assume that they have a resultant or else that three or more of them are in equilibrium.

214. The resultant of *any number* of parallel forces is parallel to the forces and equal to their algebraic sum.

Let P, Q, R, S be the parallel forces taken with their proper signs. Then the two forces P, Q have a resultant $P+Q$ parallel to them; the forces $P+Q$ and R have a resultant $P+Q+R$ parallel to them, which is therefore the resultant of the *three* forces P, Q, R; the forces $P+Q+R$ and S have a resultant $P+Q+R+S$ parallel to them, which is therefore the resultant of the *four* forces P, Q, R, S, and so on.

[This is true whether the forces are like or unlike, provided they are taken with positive or negative signs, as in § 209.]

It has been proved in § 206 that the sum of the moments of two *intersecting* forces about any point is equal to the

* Hence a *couple* in Statics does not simply mean "two of anything" as in ordinary language.

moment of their resultant. Hence it now remains to extend the theorem to the case when the forces are parallel.

215. The algebraic sum of the moments of two parallel forces (not forming a couple) about any point in their plane is equal to the moment of their resultant about that point.

(Varignon's Theorem for Parallel Forces.)

Let P, Q be the two parallel forces, R their resultant, O any point in their plane. Through O draw a straight line perpendicular to the forces P, Q, R and cutting them in A, B, C respectively. In this straight line apply two equal and opposite forces F, $-F$. Let S, T be the forces compounded of P, F and Q, $-F$ respectively; then R, the resultant of parallel forces P, Q, is also the resultant of the forces S, T. Now the forces S, T intersect one another (since P, Q do not form a couple); therefore, by § 206,

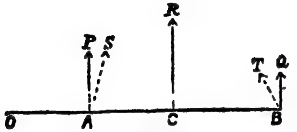

Fig. 94.

moment of R about O = algebraic sum of moments of S, T.

Also, P, Q are the resolved parts of S, T perpendicular to OB; therefore, by § 204,

$$\text{moment of } S \text{ about } \mathrm{O} = P \times \mathrm{OA} = \text{moment of } P,$$
$$\text{moment of } T \text{ about } \mathrm{O} = Q \times \mathrm{OB} = \text{moment of } Q.$$

Hence moment of R about O = algebraic sum of moments of P, Q about O,

or $$R \times \mathrm{OC} = P \times \mathrm{OA} + Q \times \mathrm{OB}, \text{ algebraically.}$$

This relation is sometimes called the **Equation of Moments.**

Alternative Proof.—With the notation of Fig. 94, let P, Q be *like* forces. Then

Algebraic sum of moments of P, Q about O

$$\begin{aligned} &= P \times \mathrm{OA} + Q \times \mathrm{OB} \\ &= P(\mathrm{OC} - \mathrm{AC}) + Q(\mathrm{OC} + \mathrm{CB}) \\ &= (P + Q)\,\mathrm{OC} + (Q.\mathrm{CB} - P.\mathrm{AC}) \\ &= (P + Q)\,\mathrm{OC} + 0 \ (\S\ 210) = R \times \mathrm{OC} \\ &= \text{moment of } R \text{ about } \mathrm{O}. \end{aligned}$$

A similar proof applies to unlike forces or to the case where **O** is between the forces, provided that the moments are subject to the usual conventions regarding signs. The student is *strongly* recommended to work out these cases as exercises.

COR. When three parallel forces are in equilibrium, the algebraic sum of their moments about any point in their plane is zero.

For each force is equal and opposite to the resultant of the other two; therefore its moment is equal and opposite to the moment of that resultant, *i.e.* to the sum of the moments of the other two forces.

216. Generalisation.—*If any number of forces act on a body in one plane, the algebraic sum of their moments about any point in that plane is equal to the moment of their resultant.*

Consider two of the forces. Whether these intersect or are parallel, algebraic sum of their moments about **O** = moment of their resultant.

Combine this resultant with a third force; then,
algebraic sum of moments of the three

= moment of third force + moment of resultant of first two
= moment of resultant of all three forces,

and so on, till all the forces have been compounded together.

COR. 1. If any number of forces acting on a body in one plane are in equilibrium, the algebraic sum of their moments about any point in the plane is zero.

For their resultant is zero; therefore its moment about any point is zero, and therefore, by above, the sum of all the moments is zero.

COR. 2. If the algebraic sum of the moments of any number of forces about a point **O** in their plane vanishes, then *either* R must be zero, or the *arm* of R must be zero. That is *either* the forces are in equilibrium *or* they have a resultant passing through **O**.

Examples.—(1) To find the resultant of two like forces of 5 lb. and 4 lb. applied at the ends of a rod 3 ft. long perpendicular to its length.

Let **AB** be the rod, **C** the point at which the resultant cuts it. (Fig. 94 may be used, taking $P = 5$ lb. and $Q = 4$ lb.)

Since the forces are like, their resultant $= 4 + 5$ lb. $= 9$ lb.

Take moments about **A**. Then the moment of 9 lb. at **C** equals the sum of the moments of 5 lb. at **A** and 4 lb. at **B**, whence, taking a foot as the unit of length,

$$9 \times AC = 5 \times 0 + 4 \times AB = 0 + 4 \times 3 = 12;$$

$$\therefore \quad AC = \tfrac{12}{9} \text{ ft.} = 1\tfrac{1}{3} \text{ ft.} = 16 \text{ in.}$$

Therefore the resultant acts at a distance of 16 in. from the 5 lb. force.

(2) To find the resultant of two *unlike* forces of 5 lb. and 4 lb. applied at points distant 3 ft. apart.

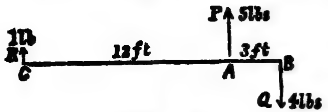

Fig. 95.

Let A, B, C be the points of application of the forces and their resultant.

Since the forces are unlike, their resultant = 5 − 4 lb. = 1 lb.

Taking moments about C, the equation of moments gives

$$P \times \mathrm{CA} - Q \times \mathrm{CB} = 0, \quad \text{or} \quad 5\mathrm{CA} = 4\mathrm{CB};$$

whence, if CA $= x$, $5x = 4(x+3)$, giving $x = 12$;

$$\therefore \quad \mathrm{CA} = 12 \text{ ft.}, \quad \mathrm{CB} = 12 + 3 = 15 \text{ ft.}$$

Therefore the resultant acts at distances of 12 and 15 ft. from the 5-lb. and 4-lb. forces respectively.

(3) To find the parallel forces which must be applied to a bar 9 ft. long in order that their resultant may be a force of 12 lb. acting at 2 ft. distance from one end.

Let P, Q be the required forces, BC the bar, A the point at which the resultant acts.

Taking moments about C, we have

$$P \times \mathrm{CB} = 12 \times \mathrm{CA};$$

$$\therefore \quad P \times 9 = 12 \times 2 = 24;$$

$$\therefore \quad P = \tfrac{24}{9} \text{ lb.} = \tfrac{8}{3} \text{ lb.} = 2\tfrac{2}{3} \text{ lb.}$$

Also $P + Q = 12$ lb.;

$$\therefore \quad Q = 12 - 2\tfrac{2}{3} = 9\tfrac{1}{3} \text{ lb.}$$

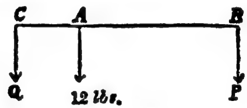

Fig. 96.

217. Conditions of Equilibrium of Three Parallel Forces.

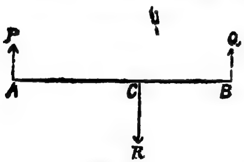

Fig. 97.

Let P, Q, R be the three forces of which P, Q are *like* forces. Let a straight line be drawn, cutting them in A, B, C.

In the figure of § 210, reverse the force R so as to make it the equilibrant instead of the resultant of P and Q. R has thus been changed in direction only, not in magnitude nor line of action. Hence we see that, when three parallel forces are in equilibrium, the middle one is equal to the sum of the two outer ones, and acts in the opposite sense. Also

$$\frac{P}{\mathrm{BC}} = \frac{Q}{\mathrm{CA}},$$

and, by a well-known theorem*, each member

$$= \frac{P+Q}{\mathsf{BC}+\mathsf{CA}} = \frac{P+Q}{\mathsf{BA}}.$$

Now R is equal and opposite to $P+Q$.

Also **AB** is equal and opposite to **BA**.

Hence the conditions of equilibrium may be written in the symmetrical form,

$$\frac{P}{\mathsf{BC}} = \frac{Q}{\mathsf{CA}} = \frac{R}{\mathsf{AB}} \quad \text{.......................(1).}$$

This result is analogous to the Triangle of Forces, and may be deduced from it. The conditions may also be stated thus:

The two extreme forces act in the same direction, and the middle force acts in the reverse direction and is equal and opposite to their sum. Also each force is proportional to the distance between the other two.

[Notice that P stands over the length which does *not* contain the letter **A**, Q, R over the lengths which do not contain **B**, **C** respectively.]

218. Experimental Verification : (*a*) Details of Experiment.—To verify, experimentally, the conditions of equilibrium of three parallel forces, take any rod, and first find **C** the centre of gravity (*i.e.* the point at which its weight acts) by making it balance on a support placed at that point. Now suspend the rod from two spring balances attached at any two points **A**, **B**, and hanging vertically; at **C** attach any known weight, and read off the spring balances.

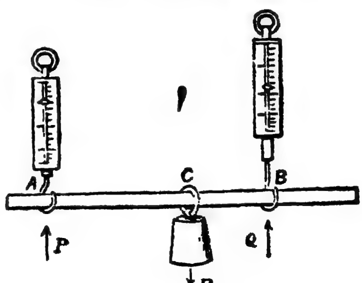

Fig. 98.

(*b*) Observed Facts—It will be found that the two readings, when added together, are exactly equal to the

* If $\frac{a}{b} = \frac{c}{d}$, each fraction $= \frac{a+c}{b+d} = \frac{\text{sum of numerators}}{\text{sum of denominators}}$.

weight of the rod together with its attached weight, both of which act at C.

Now let the distances AC, CB be measured. Then, if the experiment be carefully performed, it will be found that the readings of the two balances attached at A, B, and the total weight at C, are proportional respectively, to the lengths BC, CA, AB.

(c) DEDUCTIONS.—Hence, if R denote the total weight at C, and P, Q denote the upward thrusts exerted on the rod by the two spring balances, we shall have

$$R = -(P+Q), \quad P \times \mathrm{AC} = Q \times \mathrm{CB},$$

and

$$\frac{P}{\mathrm{BC}} = \frac{Q}{\mathrm{CA}} = \frac{R}{\mathrm{AB}};$$

agreeing with the conditions of equilibrium already found.

OBSERVATION.—The experiment can be varied by attaching a third spring balance instead of the weight at C. If the three spring balances are held horizontally, the rod resting on a table, the weight of the rod will not affect the equilibrium, and it will be found that the readings of the three spring balances attached to A, B, C are respectively proportional to BC, CA, AB, thus verifying the conditions of § 217. *See* §§ 327-9.

EXAMPLES XX.

1. Find the resultants of each of the following pairs of like parallel forces at the given distances apart, stating in each case the magnitude of this resultant and its distances from each of the two components, and illustrating by figures—

(i) 2 lb. and 3 lb., 30 in. apart.
(ii) 12 oz. and 4 oz., 8 ft. apart.
(iii) 7½ tons and 1½ tons, 48 yd. apart.
(iv) 3 lb. and 11 lb., 56 ft. apart;
(v) 4 and 16 grammes, 180 cm. apart;
(vi) 100 and 80 kilog., 45 metres apart.

2. Find, in like manner, the resultants of pairs of *unlike* parallel forces whose magnitudes and distances apart are given by the data of Ex. 1.

3. Parallel forces of 4 lb. and 8 lb. act on a bar 12 ft long, the former at one end of the bar and the latter at the middle point. Find the magnitude and point of application of the force which will balance them (i) if they are like, (ii) if they are unlike.

4. Like parallel forces of 10 lb. and 15 lb. act at the ends of a bar 5 ft. long. Find the force required to balance them and the distance of its point of application from the end of the bar at which the smaller force acts.

5. A uniform bar, 10 ft. long, balances over a rail, with a boy, weighing three times as much as the bar, hanging on to the extreme end of it. Draw a figure showing the balancing position.

6. Two parallel forces, acting in opposite directions, are to one another as 5 : 6, and the distance between their lines of action is $1\frac{1}{2}$ ft. Determine the line of action of their resultant.

7. Two unlike parallel forces, the less of which is 12 dynes, have a resultant equal to 8 dynes, which acts at a distance of 18 cm. from the smaller force. Find the line of action of the greater force.

8. A man carries a bundle at the end of a stick over his shoulder. If the distance between his hand and the bundle be kept constant, and the distance between his hand and shoulder be varied, how does the force on his shoulder change?

9. Forces P, $2P$, $3P$, $4P$ act along the sides of a square ABCD, taken in order. Find the magnitude, direction, and line of action of the resultant.

10. A man supports two weights slung on the ends of a weightless stick, 42 in. long, placed over his shoulder. Find the point of support, if one of the weights is three-fourths of the other.

11. A uniform rod, 5 ft. long and weighing 24 lb., is laid on a table with 6 in. projecting over the edge. What weight can be hung at the end of the rod before the rod will be pulled over?

12. A weight of 4 cwt. is carried by two men on a weightless rod 8 ft. long. The weight is hung at the middle of the rod; one man is $1\frac{1}{2}$ ft. from one end and the other $2\frac{1}{2}$ ft. from the other end of the rod. Find the weight borne by each.

13. Two like parallel forces P and Q act at two points in a straight line, 21 in. apart. Their resultant is a force of 7 lb. acting at a point in the line 9 in. from P, the larger of the two forces. Find the magnitude of the forces P and Q.

14. Two unlike parallel forces P and Q act at two points of a weightless rod, 4 in. apart. Their resultant is a force of 1 lb. acting at a point of the rod 3 ft. from P, the larger of the two forces. Determine the values of P and Q.

15. Show that the resultant of two unlike parallel forces acts towards the side of the greater of the two forces, and can never act between them. What happens if the forces become equal?

16. By how much must P, the greater of two like parallel forces P and Q, be diminished in order that the distance of the line of action of the resultant from P may be the same as that of the line of action of the former resultant was from Q?

17. Two like parallel forces of 4 lb. and 6 lb. respectively, act at two points in a straight line, the distance between the two points being 5 ft. How far is the point of application of their resultant moved when each force is increased by 1 lb., and in which direction?

18. Two unlike parallel forces of 4 lb. and 6 lb. respectively, act at two points in a straight line, and the distance between the points of application of the forces is 2 ft. When equal additions are made to the two forces the point of application of their resultant is moved through 2 ft. Find what force is added to each of the given forces.

19. A uniform heavy beam, 8 ft. long, rests horizontally on two supports, one at one end and the other 3 ft. from the other end. If the greatest mass that can be hung at the latter end without upsetting the beam be 20 lb., find the weight of the beam.

20. Two men carry a uniform beam 15 ft. long and weighing 120 lb. One man supports it at a distance of $1\frac{1}{2}$ ft. from one end, and the other man at a distance of $3\frac{1}{2}$ ft. from the other end. What weight does each man support?

21. A weightless rod, 15 in. long, rests horizontally with one end on the edge of a table and the other supported by a vertical string. If a weight of 6 lb. be suspended from the rod at a certain distance from the table, the tension of the string will be 4 lb. Find the force of pressure on the table and the point where the weight is attached.

CHAPTER XXI.

SYSTEMS OF PARALLEL FORCES: COUPLES.

219. Method of finding the resultant of coplanar parallel forces.

When a number of parallel forces act on a rigid body, it would of course be possible to find their resultant by compounding two of them into a single resultant, then compounding this result with a third, and so on. But if the forces all act *in the same plane*, the same thing can be done more easily by writing down the equations which express the facts that—

(i) *the* **magnitude** *of the resultant equals the algebraic sum of its components* (§ 214);

(ii) *the* **moment** *of the resultant about any point equals the algebraic sum of the moments of its components* (§ 216).

The point about which moments are taken may be chosen anywhere in the plane of the forces, but some points (very often *one* point) are generally more convenient than others. But it is important to notice that the final result is the same *whatever* point is chosen.

An outside point is generally the best to choose, as there is then no difficulty in distinguishing the signs of the various moments.

Example.—Weights of 4 lb. and 12 lb. are attached to the ends of a uniform rod, 14 ft. long, weighing 12 lb. To find the point at which their resultant acts.

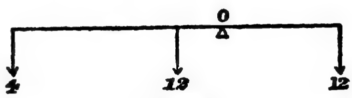

Fig. 99.

The three forces on the rod are 4 lb. and 12 lb. acting at its ends, and its weight 12 lb., which, since the rod is uniform, may be supposed to act at its middle point, 7 ft. from either end.

The *magnitude* of their resultant = 12 + 12 + 4 lb. = 28 lb.

The required point O, at which this resultant cuts the rod, may be found by taking moments about the end at which the 4 lb. acts. Let

the distance of O from this end be x ft. Then the equation of moments gives

$$28.x = 4.0 + 12.7 + 12.14 = 252.$$

$$\therefore \quad x = 9 \text{ ft.},$$

or the point O is 9 ft. from the weight of 4 lb., and 5 ft. from the other end.

220. To find the resultant of any number of given parallel forces acting in one plane.

Let $P_1, P_2, P_3, \ldots$ denote the given forces both in magnitude and algebraic sign. Draw any line $\mathbf{OA_1A_2A_3}\ldots$ at right angles to the forces. On it take any point **O**, the distances $\mathbf{OA_1}$, $\mathbf{OA_2}, \ldots$ being known.

(i) To find the magnitude of the resultant.

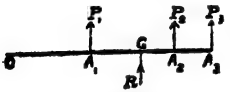

Fig. 100.

The resultant of P_1, P_2 is a parallel force of magnitude $P_1 + P_2$; the resultant of this force and P_3 is therefore $P_1 + P_2 + P_3$; and so on. Hence the final resultant is a parallel force R, such that

$$R = P_1 + P_2 + P_3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$= \text{algebraic sum of the forces.}$$

(ii) To find the position of the resultant R.

Let it cut $\mathbf{OA_1}\ldots$ in a point **G**, the position of which is required. Since the moment of the resultant is equal to the algebraic sum of the moments of the components,

$$\therefore \quad R.\mathbf{OG} = P_1.\mathbf{OA_1} + P_2.\mathbf{OA_2} + P_3.\mathbf{OA_3} + \ldots.$$

Hence $$\mathbf{OG} = \frac{P_1.\mathbf{OA_1} + P_2.\mathbf{OA_2} + P_3.\mathbf{OA_3} + \ldots}{R},$$

or $$\mathbf{OG} = \frac{P_1.\mathbf{OA_1} + P_2.\mathbf{OA_2} + P_3.\mathbf{OA_3} + \ldots}{P_1 + P_2 + P_3 + \ldots} \quad \ldots(2).$$

Equations (1), (2) determine, respectively, the magnitude and position of the resultant.

If $x_1, x_2, x_3, \ldots$ denote the known distances of the component forces from **O**, and $\bar{x}$ the required distance of their resultant; equation (2) becomes

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}\ldots\ldots\ldots\ldots\ldots\ldots(2a).$$

221. Equilibrium of a Loaded Beam Resting on Two Supports.—When a beam is loaded with given weights placed at given points, and rests in a horizontal position on two props, it is often necessary to determine the forces of pressure on the props, or, what amounts to the same thing, the reactions of the props on the rod (which are equal and opposite to them).

These reactions, together with the weights on the beam, form a system in equilibrium, and therefore

(i) The sum of the forces is zero,

(ii) The sum of their moments about any point is zero.

If we want to find the reactions one at a time, we therefore proceed as follows :—

(i) *Equate to zero the algebraic sum of the forces (including the two reactions). The equation gives the sum of the reactions.*

(ii) *Take moments about one of the props. The reaction of that prop has no moment, and therefore the equation of moments at once gives the reaction of the other prop.*

If, in any example, the thrust on one of the props should come out *negative*, it is to be inferred that the rod presses *upwards* on its support, and that the latter has to hold it down.

Examples.—(1) A uniform rod, 10 ft. long, of weight W, is supported on trestles at both ends, and a weight $5W$ is placed on it 4 ft. from one end. To find the forces of pressure on the trestles.

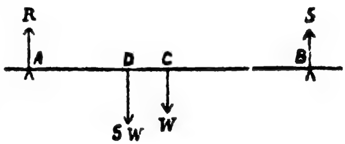

Fig. 101.

Let **AB** be the rod, **C** the middle point at which its weight W may be supposed to act, **D** the point of application of the weight $5W$.

Let R, S be the required forces of pressure at **A**, **B**.

Then, by equating forces, we have

$$R + S = 5W + W = 6W \text{ ....} \quad \text{(i)}.$$

Taking moments about **A**, we have

$$S \times 10 = W \times 5 + 5W \times 4 \text{.......................} \quad \text{(ii)}.$$

From (ii), $S = \frac{5}{2}W$. Hence, by (i), $R = \frac{7}{2}W$.

(2) A uniform rod, 12 ft. long, weighing 20 lb., has weights of 12 lb. and 4 lb. attached to its ends, and 8 lb. attached at a distance of 4 ft. from the 4-lb. weight. It is placed on two props, 8 ft. apart, so that the end with the 4-lb. weight projects 1 ft. To find the reactions of the props.

In Fig. 102, let **M**, **N** be the props, and let their reactions be R lb. and S lb. respectively.

Let **G** be the middle point of the rod at which its weight (20 lb.) acts.

(i) Here, since the sum of the upward force is equal and opposite to the sum of the downward ones (or the algebraic sum of the forces is zero),

$$\therefore \quad R + S = 12 + 20 + 8 + 4 = 44.$$

(ii) Taking moments about M, we have

$$S.\mathrm{MN} = 12.\mathrm{MB} + 20.\mathrm{MG} + 8.\mathrm{MC} - 4.\mathrm{AM},$$

or
$$8S = 12.11 + 20.5 + 8.3 - 4.1$$
$$= 132 + 100 + 24 - 4 = 252;$$

whence $S = 31\frac{1}{2}$ lb.

Therefore $R = 44 - 31\frac{1}{2}$ lb. $= 12\frac{1}{2}$ lb.

Hence the thrusts on the props are $12\frac{1}{2}$ lb. and $31\frac{1}{2}$ lb.

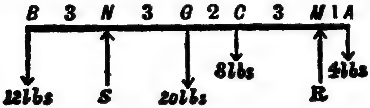

Fig. 102.

[If we had taken moments about N, we should have had

$$R.\mathrm{NM} = 4.\mathrm{NA} + 8.\mathrm{NC} + 20.\mathrm{NG} - 12.\mathrm{BN},$$
$$8R = 4.9 + 8.5 + 3.20 - 12.3$$
$$= 36 + 40 + 60 - 36 = 100;$$

whence $R = 12\frac{1}{2}$ lb.,

agreeing with the value just found and affording a test of the accuracy of the calculation.]

222. Definition.—A couple consists of two equal forces acting in opposite directions along two parallel straight lines. *A couple cannot keep a body in equilibrium*, for it tends to rotate the body: the points of application of the two forces of the couple tending to move in opposite directions (§ 193). Moreover, the proof that two parallel forces have a single resultant fails for the case of a couple (§ 213).

Examples of couples.—In winding a clock we apply a *couple* to the key, for we do not try to make it move to one side or the other, but simply turn it round. To spin a small top between the finger and thumb, we apply a *couple* to it by moving the finger and thumb sharply in opposite directions. To open a door we apply a *couple* to the handle.

223. Definitions.—The arm of a couple is the perpendicular distance (AB, Fig. 103) between the lines of action of its two components (*i.e.* the two forces forming the couple).

The moment of a couple is the algebraic sum of the moments of its two components about any point in their plane.

The following is the fundamental property of couples :—

224. The moment of a couple is the same about all points in its plane.

Let the couple consist of two equal and opposite forces $P, -P$ at B and A. Let O be any point in their plane. Draw OAB perpendicular to the forces.

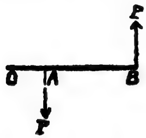

Fig. 103.

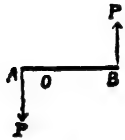

Fig. 104.

Then, if O does not lie between A and B, as in Fig. 103, algebraic sum of moments of forces $= P.\mathrm{OB} - P.\mathrm{OA}$

$$= P(\mathrm{OB} - \mathrm{OA}) = P.\mathrm{AB}.$$

If O lies between A and B, as in Fig. 104, we have

algebraic sum of moments $= P.\mathrm{OB} + P.\mathrm{AO}$

$$= P(\mathrm{AO} + \mathrm{OB}) = P.\mathrm{AB}.$$

Hence the moment of a couple about O *is independent of the position of* O *and is equal to the product* $P.\mathrm{AB}$.

For an instructive experiment see § 331.

225. Alternative expressions for the moment of a couple.

The moment of a couple may therefore be defined as—

(i) *The product of the measure of either force into the arm of the couple.*

(ii) *The moment of either of the two forces about any point in the line of action of the other force.* For

moment $= P \times \mathrm{AB} =$ moment about A of P acting at B.

226. A couple cannot be replaced by a single force.

For the moment of a single force about any point on its line of action is zero (§ 201). But the moment of a couple about every point in its plane is a constant quantity, differing from zero.

Hence a couple cannot have a resultant.

***227. Two couples in the same plane whose moments are equal and opposite will balance one another.**

Let one of the couples consist of two forces P, $-P$ acting on the arm **AB**, and let the other couple consist of forces Q, $-Q$ acting on the arm **CD**.

Suppose that the forces P and Q are not parallel. Then the lines of action of the four forces must form a parallelogram *abcd*.

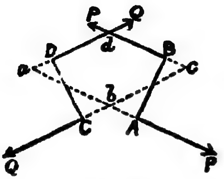

Fig. 105.

Since the moments of the couples are equal and opposite about any point in the plane, therefore the moments about b of P and Q acting along *ad* and *cd* are equal and opposite, and therefore the moment of the resultant of these forces about b is zero (§ 206).

Therefore the resultant of P and Q at d must pass through b, and therefore it acts along *bd*.

Similarly the resultant of $-P$, $-Q$ at b acts along *db*.

But the latter resultant is equal and opposite to the former, for the two components of the latter are respectively equal and opposite to those of the former.

Hence the two resultants balance each other, and therefore the four forces forming the two couples are in equilibrium.

COR. *Principle of transmission of couples.*—A couple may be replaced by any other couple of equal moment.

Fig. 106.

From this result we also see that a couple has no particular position of application, but that it may be shifted anywhere in its plane without altering its statical effect. The effect of the couple depends, therefore, only on its moment and the plane in which it acts.

***228. A force, acting at any point of a body, is equivalent to an equal and parallel force acting at any other point together with a couple.**

Let P (Fig. 106) be a force acting at a point **O**; to show that it is equivalent to an equal and parallel force P, acting at any other point **A**, together with a couple.

Introduce two equal and opposite forces P, $-P$, acting at **A**, numerically equal and parallel to the force P at **O**.

The effect of these forces will be to neutralise one another.

But the forces P at **O** and $-P$ at **A** form a couple whose moment (about **A**) is equal to the moment of the original force P about **A**.

Thus the original force P at **O** is equivalent to this couple and a parallel and equal force P at **A**.

NOTE. *Any number of coplanar couples are equivalent to a single couple whose moment is the algebraic sum of their moments.*

EXAMPLES XXI.

1. A uniform straight rod, 16 in. long, is acted on by five like parallel forces of 2, 3, 4, 5, and 6 lb. respectively, acting at intervals of 4 in. Find the point of application of their resultant.

2. Weights of 16 lb. and 9 lb. are attached to the ends of a uniform straight rod 20 in. long and weighing 15 lb. Where must the rod be supported in order that it may balance, and what is the pressure on the point of support?

3. A uniform beam, $4\frac{1}{2}$ ft. long, rests horizontally on two props placed under its extremities, and the pressure of the beam on each prop is 5 lb. Where must a weight of 54 lb. be placed so that the whole pressure on one of the props may be 12 lb.?

4. A uniform beam, 14 ft. long and weighing 120 lb., is attached to two props, one of which is 3 ft. and the other 5 ft. from its centre Calculate the forces on the props when a weight of 100 lb. is placed first on one end and then on the other end of the beam.

5. To a uniform rod weighing 28 lb., weights of 6, 8, 10, and 12 lb. are attached at equal distances, the weight of 6 lb. being at the middle point of the rod. Find the point about which the rod will balance.

6. Forces of 3, 5, 7, and 9 lb. respectively, act vertically downwards at equal distances of 4 in. along a horizontal line. Find the point of application of their resultant.

7. Weights of 1, 2, 3, 4, and 5 lb. are hung at equal intervals of 6 in. along a weightless rod 24 in. long. Where must the rod be supported in order that it may remain horizontal?

8. A uniform beam, 24 ft. long and weighing 200 lb., is supported on two props, one 6 ft. from one end and the other 9 ft. from the other end of the beam. Calculate the pressure on each prop when a man weighing 180 lb. stands as near this latter end as he can without upsetting the beam, and find his position.

9. A uniform horizontal bar, 16 ft. long and weighing 20 lb., is supported at both ends, while 4 ft. from one end a weight of 64 lb. is hung. Find the reactions of the supports.

10. A uniform beam, 12 ft. long and weighing 56 lb., rests on and is fastened to two props 5 ft. apart, one of which is 3 ft. from one end of the beam. A load of 35 lb. is placed (*a*) on the middle of the beam, (*b*) at the end nearest a prop, (*c*) at the end furthest from a prop. Find the weight each prop has to bear in each case.

11. A see-saw consists of a plank 14 ft. long, weighing 80 lb., and two boys, whose weights are 60 lb. and 100 lb., sit on the plank at a distance of 1 ft. from each end. Find where the plank must be supported in order that it may balance.

12. Two men carry a load of 1 cwt. suspended from a horizontal uniform pole 12 ft. long, whose weight is 20 lb., and whose ends rest on their shoulders. Where must the load be suspended in order that one of the men may bear 94 lb. of the whole weight?

13. A uniform rod, 3 ft. long and weighing 14 lb., rests on a horizontal table with one end projecting 6 in. over the edge. Find the greatest weight which can be hung on the end without making the rod topple over.

*14. Find the resultant of a force of 5 lb. and a couple whose arm is $2\frac{1}{2}$ ft. long and whose forces are each 4 lb. State its position clearly with reference to the 5 lb. force.

*15. Find the resultant of a force and a couple acting in the same plane on a rigid body.

16. Show that a couple has no particular point of application, but may be shifted about anywhere in the same plane without disturbing the equilibrium of a body to which it is applied. Is this true of a force? Explain the difference, if any.

*17. P and Q are like parallel forces whose lines of action are at a distance p apart; another unlike parallel force $P + Q$ acts in the same plane with P and Q at a distance q from the force P. Find the moment of the couple formed by the three forces (i) when $P + Q$ and Q are on the same side of P, (ii) when they are on opposite sides.

18. Two forces of 10 lb. each act from **A** to **B** and from **C** to **D**, respectively, along the sides **AB**, **CD** of a square **ABCD**; a third force of 12 lb. acts along the diagonal **CA**. Find their resultant, and show in a diagram exactly how it acts.

19. Three forces act along the sides of a triangle **ABC**, taken in order, and are proportional to the sides along which they act, their magnitudes being $P.\mathbf{AB}$, $P.\mathbf{BC}$, $P.\mathbf{CA}$. Find the moment of the couple formed by the forces.

*20. P and Q are like parallel forces, and an unlike parallel force $P + Q$ acts in the same plane at distances p and q respectively, from the two former, and between them. Find the moment of the couple formed by the three forces.

21. Forces of 4, 6, and 4 lb. act along the sides AB, CB, CD of a rectangle ABCD; and the sides AB, BC are 8 and 12 in. long respectively. Find how another force of 6 lb. must act in order that the four forces may produce equilibrium.

EXAMINATION PAPER XI.

1. What is meant by the *moment* of a force about a point? Explain clearly how it can be represented geometrically.

2. Equal forces, each of 10 lb., act along the sides BC, CA, AB of a triangle ABC. If the lengths of these sides be 3, 4, and 5 ft., respectively, find the moment of each force about the opposite angular point.

3. Show how to find the resultant of two unlike parallel forces.

4. A uniform bar, 20 ft. long and weighing 300 lb., rests in a horizontal position on two supports, one of which is 4 ft. from one end of the bar, while the other is 1 ft. from the other end. Find the force of pressure on each support.

5. Show that the algebraic sum of the moments of two parallel forces (not forming a couple) about any point in their plane is equal to the moment of their resultant about that point.

6. How would you verify, experimentally, the conditions of equilibrium of three parallel forces acting upon a rigid body.

7. What is meant by a *couple* in Mechanics? Show that the moment of a couple is the same about all points in its plane.

8. Show that a force acting at any point of a body is equivalent to an equal and parallel force acting at any other point and a couple about that point.

9. State (without proof) the conditions which must be satisfied in order that two couples may balance.

10. If six forces are represented, fully, by the sides of a regular hexagon, taken in order, show that they constitute a couple, and obtain a measure of its moment.

CHAPTER XXII.

MACHINES: THE LEVER: THE WHEEL AND AXLE.

229. A **machine** in Mechanics means any contrivance in which a force applied at one point is made to raise a weight or overcome a resisting force acting at another point. The former is called the **effort** or **power**, the latter the **resistance** or **weight.***

In what follows, the effort will be denoted by P, and the resistance by either Q or W, the letter W being generally used when the resistance is a heavy weight which the machine has to lift.

Machines are used for the following purposes :—

(1) To enable a person to raise weights or overcome resistances so great that the effort he is capable of exerting would be insufficient without the use of a machine.

Example.—A truck drawn up an inclined plane to any required height when it is too heavy to be lifted bodily off the ground.

(2) To enable the motion imparted to one point of a machine to produce a much more rapid motion at some other point.

Example.—A bicycle, or a winnowing machine.

(3) To enable the effort to be applied at a more convenient point, or in a more convenient direction, than that in which the resistance acts.

Example.—A poker used to stir the fire, a rope and pulley for raising a bucket from a well.

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* The terms "power" and "weight" are used in the older books on Mechanics, and still sometimes occur in examination papers; but "power" also signifies "rate of working," such as horse-power, and machines are often used in overcoming resistances other than those due to gravity or "weight."

In every case a machine must be capable of *moving* the point of application of the resistance, *i.e.* of *doing work* against the resistance, and the Principle of Work teaches us that, in all cases, an equal amount of work must be done by the effort in moving the machine; in other words, *we must put as much work into the machine as we want to get out of it.*

230. The Mechanical Powers.—The simplest forms of machines are called the **mechanical powers**, and it is usual to distinguish the following six forms of them :—

The inclined plane.	[Chap. XVII.]
The wedge.	[Chap. XVIII.]
The lever.	
The wheel and axle, and windlass.	
The pulley and systems of pulleys	[Chap. XXIII.]
The screw.	[Chap. XXIII.]

In every case we suppose these machines to be devoid of friction, and in Statics we are chiefly concerned with finding the relations between the effort and the resistance, when there is equilibrium.

231. Mechanical Advantage: DEFINITION.—The **mechanical advantage** is the number which expresses what multiple the resistance is of the effort, *i.e.*

$$\text{mechanical advantage} = \frac{\text{resistance}}{\text{effort}} = \frac{Q}{P} \text{ or } \frac{W}{P}.$$

Consider, for example, a smooth inclined plane at a slope of, say, 1 in 20. By applying a force of 1 cwt. along the plane, it is possible to draw a weight of 20 cwt. or 1 ton up the plane, and if the plane be long enough, this weight may be raised to any desired height. The resistance to be overcome is that due to gravity, viz. the weight of 20 cwt. It is therefore twenty times the effort, and we say that *the mechanical advantage is* 20. Generally, if the effort acts up a smooth incline of 1 in n, the mechanical advantage is n.

232. The lever is a rigid bar capable of turning freely about a fixed point of support. This point is called the **fulcrum.** The *effort* is a force applied at any point of the lever, so as to turn it about the fulcrum, and thus to raise a weight or overcome a resistance applied at any other point.

The lever is most often a straight rod, the two arms therefore being in one straight line, and the effort and resistance generally act perpendicular to the arms. But these are mere matters of convenience.

In theoretical calculations we neglect the thickness of the lever and most frequently assume it to be without weight.

For experiments on levers see § 332.

233. To find the mechanical advantage of the lever when the forces act perpendicular to the arms.—Let **C** be the fulcrum, **CA, CB** the arms, and let a force P applied at **A** perpendicular to **CA** support a resistance Q applied at **B** perpendicular to

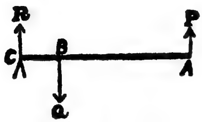


Fig. 107.

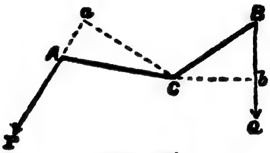


Fig. 108.

CB. Then the condition of equilibrium requires the moments of P and Q about **C** to be equal and opposite, and therefore (Fig. 107)

$$P \times \mathbf{CA} = Q \times \mathbf{CB}.$$

This condition may be written

$$\frac{Q}{P} = \frac{\mathbf{CA}}{\mathbf{CB}}.$$

$$\therefore \text{ mechanical advantage } \frac{Q}{P} = \frac{\text{arm of effort}}{\text{arm of resistance}} \quad (1);$$

or *the effort and resistance are inversely proportional to the arms on which they act.*

Hence, by applying the effort at the end of a long arm and the resistance very near the fulcrum, the mechanical advantage may be made very great, and a man may raise a weight may times greater than he could lift bodily off the ground.

Example.—Thus, by exerting a force of 1 lb. at a distance of 2 ft. from the fulcrum, we can lift a weight of 2 lb. applied at a distance of 1 ft. from the fulcrum, or a weight of 24 lb. applied an inch away from the fulcrum.

234. Mechanical Advantage of Levers in General.—Where P, Q do not act perpendicular to CA, CB, we must drop Ca, Cb perpendicular on their lines of action (Fig. 108).

Taking moments about C, the condition of equilibrium now becomes $$P \times Ca = Q \times Cb.$$

$$\therefore \quad \text{mechanical advantage } \frac{Q}{P} = \frac{Ca}{Cb}$$

$$= \frac{\text{perp. dist. of effort from fulcrum}}{\text{perp. dist. of resistance from fulcrum}}.$$

Example.—A straight lever, whose arms are 2 ft. and 3 ft. long, rests at an inclination of 60° to the horizon, and a weight of 18 lb. hangs vertically from its shorter arm. To find the horizontal force which must be applied to its longer arm in order to balance.

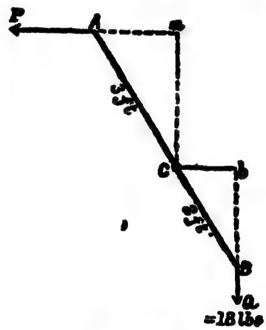


Fig. 109.

Let ACB be the lever (Fig. 109). Then, if P, Q denote the effort and weight, the directions of these forces make acute angles of 60° and 30° respectively with AB; therefore CAa, BCb are semi-equilateral triangles.

Hence $Ca = \frac{1}{2}\sqrt{3} \times 3$

and $Cb = \frac{1}{2} \times 2 = 1.$

Therefore the Lever Equation gives

$$P \times \tfrac{1}{2}\sqrt{3} \times 3 = Q \times 1 = 18$$

whence required force P

$$= \frac{18 \times 2}{3\sqrt{3}} = \frac{18 \times 2\sqrt{3}}{9} = 4\sqrt{3} \text{ lb.}$$

235. Conditions of Equilibrium of a Heavy Lever.—When the weight of a lever itself has to be taken into account, the condition of equilibrium may be found as in other cases by taking moments about the fulcrum.

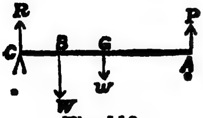


Fig. 110.

Example.—In a uniform heavy lever, whose fulcrum is at one end, a force of 3 lb. at the other end will lift a load of 8 lb., and a force of 4 lb. will lift a load of 12 lb. To find the weight of the lever and the point of application of the load.

Let w be the weight of the lever acting at its middle point G. Let the load be placed at B, and let CA $= a$, CB $= x$. Then, by taking moments about C, we have

$$3\times a = w\times\tfrac{1}{2}a + 8\times x,$$
$$4\times a = w\times\tfrac{1}{2}a + 12\times x.$$

Subtracting, we have $$a = 4x;$$

$$\therefore\ x = \tfrac{1}{4}a, \text{ or CB} = \tfrac{1}{4}\text{CA};$$

whence, by substitution, $$w = 2 \text{ lb.}$$

That is, the weight of the lever is 2 lb. and the distance of the load from the fulcrum is $\frac{1}{4}$ the distance of the effort.

236. The Three Classes of Lever.—*Straight* levers are sometimes divided into three classes, according to the relative positions of A, B, C, the points of application of the effort and resistance and the fulcrum. We shall suppose the effort and resistance to be parallel. In considering the different cases, it is convenient to use the symmetrical conditions of equilibrium of three parallel forces. If R be the reaction of the fulcrum on the lever, we have, therefore, by § 217,

$$\frac{P}{\text{BC}} = \frac{Q}{\text{CA}} = \frac{R}{\text{AB}},$$

the *middle* force acting in the opposite direction to the outer ones and being equal to their sum.

The thrust of the lever against its support at C is a force equal and opposite to R.

For convenience we shall often suppose that the resistance Q is a weight which acts downwards.

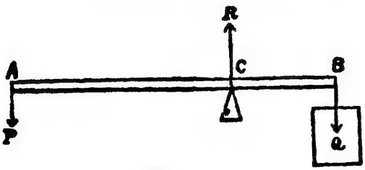


Fig. 111.

237. A lever of the first class (Fig. 111) is one in which the fulcrum is placed between the effort and resistance.

Here R is the middle force; therefore P, Q act in the same direction, and R in the opposite direction, also

$$R = P + Q.$$

Thus, in order to lift a weight, the effort must be applied *downwards*, and the reaction acts *upwards* (so that the lever presses *downwards* on the fulcrum).

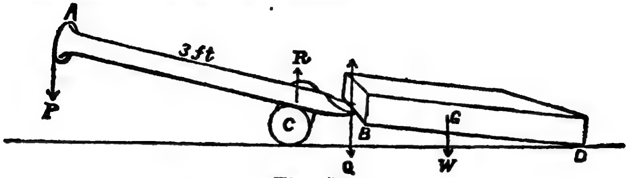


Fig. 112.

In this lever, **BC** may be either greater or less than or equal to **CA**. Therefore the effort may either be greater or less than or equal to the weight which it has to lift; so that *the mechanical advantage may be less or greater than, or equal to unity.*

Examples of this Class.—The handle of a pump; a crowbar when it rests on a block in front of the weight to be lifted and not with its end on the ground (Fig. 112); a poker, used to raise the coals in a grate, a bar of which is the fulcrum; a spade, in digging; a see-saw.

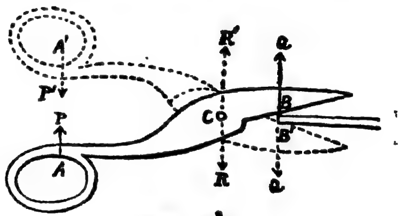


Fig. 113.

Double Lever.—A pair of scissors. Fig. 113 shows the forces acting on the arms of the scissors, those on the dotted arm being accented.

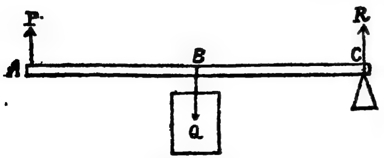


Fig. 114.

238. A lever of the second class (Fig. 114) is one in which the resistance is placed between the fulcrum and the effort.

Here Q is the middle force; therefore P and R act in the same direction, and Q in the opposite direction, also

$$Q = P + R, \quad R = Q - P.$$

Thus, in order to lift a weight, the effort must be applied *upwards*, and the reaction of the fulcrum also acts *upwards* (so that the lever presses *downwards* on the fulcrum).

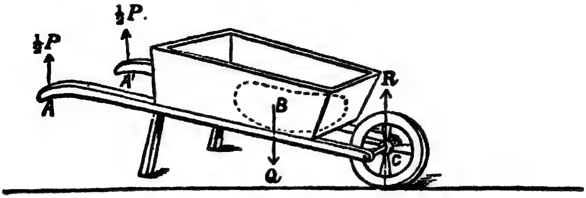


Fig. 115.

Since **CA** > **CB**, the effort is less than the weight; so that *the mechanical advantage is always greater than unity.*

Examples of this Class.—A wheelbarrow (Fig. 115), the fulcrum being where the wheel touches the ground; a crowbar, when the lower end rests on the ground.

NOTE. In the wheelbarrow half the effort is applied at each handle.

Double Lever.—A pair of nut-crackers, the forces on the two arms being shown in Fig. 116.

*239. **An oar is often called a lever of the second class.** It cannot be strictly said to belong to either class. If the boat were kept at rest and the oar used to scoop the water backwards, it would be a lever of the first class, with the rowlock as fulcrum. When the boat moves forwards instead of the water moving backwards, the relative motion and the relation between the effort applied to the handle and the resistance of the water are the same as before, *and can be correctly found by treating the oar as a lever of the first class.*

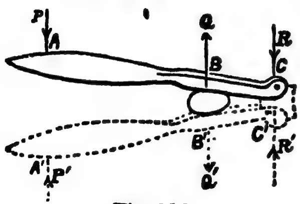


Fig. 116.

The next class of levers is rarely used, for, since the mechanical advantage is less than unity, a greater effort is necessary to overcome the resistance than if no lever were used at all.

240. A lever of the third class (Fig. 117) is one in which the effort is applied between the fulcrum and the resistance.

Here P is the middle force; therefore Q, R act in the same direction, and the effort P acts in the opposite direction; also

$$P = Q + R,$$

or

$$R = P - Q.$$

Thus, in order to lift a weight, the effort must be applied *upwards*, and the reaction of the fulcrum acts *downwards*; so that the lever presses *upwards* on the fulcrum.

Since $CB > CA$, the effort is greater than the weight; so that *the mechanical advantage is always less than unity*.

Fig. 117.

This is sometimes expressed by saying that there is *mechanical disadvantage*.

Examples of this Class.—The treadle of a turning-lathe or scissors-grinding machine; the human arm.

Double Lever.—A pair of tongs (Fig. 118).

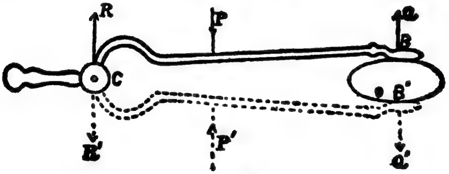


Fig. 118.

241. The wheel and axle are two cylindrical rollers joined together with a common axis terminating in two pivots about which they can turn freely. The larger roller is called the wheel, and the smaller the axle. Both the wheel and the axle have ropes coiled round them in opposite directions. The rope on the axle supports the weight, and the effort is applied by pulling the rope attached to the wheel. As the rope round the wheel unwinds, that round the axle winds up and raises the weight. Fig. 120 shows an end view of the arrangement.

242. Mechanical Advantage of the Wheel and Axle.—The condition of equilibrium is the same as if the strings were really in a vertical plane perpendicular to the common axis as they appear in Fig. 120, and therefore the moments of the effort P and weight W about the axis are equal and opposite. Here, if a denotes the radius of the wheel, and b that of the axle, then a, b are the arms on which P and W act, and therefore

$$Pa = Wb;$$

i.e. effort × rad. of wheel = weight × rad. of axle.

$$\therefore \quad \text{mechanical advantage} = \frac{W}{P} = \frac{a}{b} = \frac{\text{rad. of wheel}}{\text{rad. of axle}}.$$

By making the wheel larger and the axle smaller, the mechanical advantage will be increased.

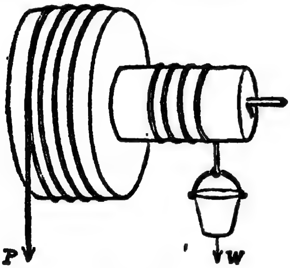


Fig. 119.

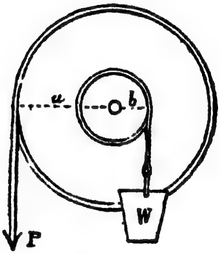


Fig. 120.

The wheel and axle is thus really equivalent to a lever whose arms are the radii of the wheel and the axle. But the lever can only be used for raising weights through short distances; the wheel and axle will lift them to any desired height.

[Instead of the rope being coiled round the wheel, an endless rope may be used, passing round a groove cut in the rim of the wheel, as in a common roller-blind, provided proper precautions are taken to prevent the rope from slipping round in the groove.]

243. The windlass (Fig. 121) is a modification of the wheel and axle, the only difference being that the effort is applied by turning a handle **AH**, at the end of an arm **CA**. It is commonly used for raising buckets of water from a well, or

earth from a shaft. [An improved form has two buckets so arranged that the empty one goes down as the full one comes up.]

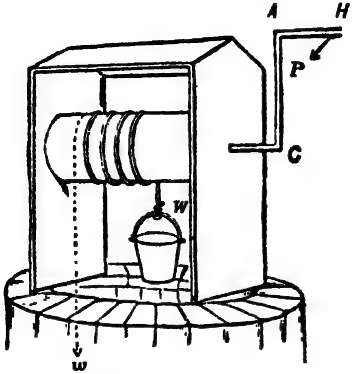


Fig. 121.

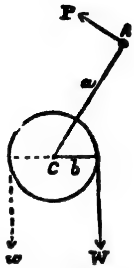


Fig. 122.

244. Mechanical Advantage.—If a is the length of the arm **CA**, the equation of moments gives, as before,

$$Pa = Wb;$$

and $\therefore$ mech. advantage $= \dfrac{a}{b} = \dfrac{\text{length of arm}}{\text{radius of axle}}$,

the length of the arm taking the place of the radius of the wheel.

[If there are two buckets, and the total ascending and descending weights are W and w, we shall have, by taking moments,

$$Pa = Wb - wb = (W - w)\,b.]$$

Example.—The axle of a windlass is 8 in. in diameter, and carries two buckets of equal weight on opposite sides. To find the force which must be applied to a handle, whose arm is 2 ft., to raise 3 gallons of water, a gallon weighing 10 lb.

Let P be the force, w the weight of each bucket. Then, since the radius of the axle is 4 in., the arm of the handle 24 in., and the total weights on the two sides w and $w + 30$ lb., we have, by moments,

$$P\times 24 + w\times 4 = (w + 30)\times 4;$$

whence $P = 5$ lb.

[Notice that the weights of the two buckets balance each other.]

245. **The capstan** (Fig. 123) used on board ship is exactly similar in principle, but the barrel turns on a vertical axis and is worked by one or more men walking round and pushing a number of horizontal projecting arms (called handspikes). Here the moment of the pull of the rope is equal to the *sum* of the moments of the forces exerted by the men.

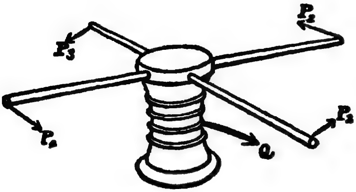


Fig. 123.

Example.—The barrel of a capstan is 3 ft. in diameter, and is worked by four men exerting forces of 45, 52, 63, and 64 lb. on arms each $7\frac{1}{2}$ ft. long. The rope passing round the barrel is fastened to a pier. To find the force drawing the ship towards the pier.

Let Q lb. be the required force exerted by the rope. Then, since the radius of the barrel is $1\frac{1}{2}$ ft., we have, by taking moments,

$$Q \times 1\tfrac{1}{2} = (45 + 52 + 63 + 64) \times 7\tfrac{1}{2};$$

$$\therefore \quad Q = 224 \times 5 \text{ lb.} = 1120 \text{ lb.} = \tfrac{1}{2} \text{ ton.}$$

246. **Principle of Work for any Machine.**—Although a small effort may be made to overcome a very large resistance with a machine, the Principle of Work, or Principle of Conservation of Energy, holds good in every case, and asserts that the work done by the effort is always equal to the work done by the machine against the weight or resistance.

Hence no work is gained or lost by the use of a frictionless machine.

For instance, if, in any machine, a force of 1 lb. supports a weight of 10 lb., the former force will have to move its point of application through 10 ft. to raise the weight through 1 ft.

This is sometimes expressed by saying that "what is gained in power is lost in speed." In more accurate language, mechanical advantage is always obtained at the expense of a proportionate disadvantage in diminished speed.

Conversely, where increased speed is obtained by means of a machine, this is only attained at the expense of mechanical disadvantage.

Example.—The arms of a lever are 3 ft. and 1 ft. To find the force on the longer arm and the work done in raising a weight of 12 lb. through 1 in., and to verify the Principle of Work.

Let P be the required force. Then, by taking moments about the fulcrum,

$$P \times 3 = 12 \times 1, \quad \text{whence} \quad P = 4 \text{ lb.}$$

Let the lever be turned about the fulcrum. Then the points furthest from the fulcrum will move over the greater distances; and, by drawing a figure with the lever in two positions, it is easy to see, or to prove, by similar triangles, that distances moved by different points are proportional to their distances from the fulcrum. Thus, if the end of the shorter arm moves 1 in., that of the longer arm will move 3 in.

Now work required to lift 12 lb. through 1 in.

$$= 12 \times \tfrac{1}{12} = 1 \text{ ft.-lb.}$$

Work done by P, or 4 lb., in moving its point of application through 3 in. $= 4 \times \tfrac{3}{12} = 1$ ft.-lb.

$\therefore$ work done by P = work required to raise weight.

Therefore the Principle of Work is true in this case.

***247. Principle of Work for the Wheel and Axle.**—Let a, b be the radii of the wheel and axle, and let them be rotated through one complete turn. Then a length of rope equal to the circumference of the wheel, or $2\pi a$, uncoils from off the wheel, and a length equal to the circumference of the axle, or $2\pi b$, coils round the axle.

Hence, work done by $P = P \times 2\pi a$,

and work done against $W = W \times 2\pi b$.

(i) *If we assume the equation of moments*

$$P \times a = W \times b,$$

then $$P \times 2\pi a = W \times 2\pi b,$$

or work done by P = work done against W,

verifying the truth of the Principle of Work for the wheel and axle.

(ii) Conversely, *if we assume the Principle of Work to be true*, then $P \times a = W \times b$,

verifying the relation between the effort and resistance, which is otherwise obtainable from the equation of moments.

***248. To find the mechanical advantage of any machine from the Principle of Work.**

We can also see that the mechanical advantage or the condition of equilibrium of a machine working without friction can very easily be found by means of the Principle of Work when they *cannot* be easily found by other methods.

Example.—If, by moving a handle through 1 ft., a weight of 1 cwt. is raised through 1 in., to find the force that must be applied to the handle.

Let P be the force. Since the works of P and the resistance are equal and opposite,

$$\therefore \quad P \times 1 = 112 \times \tfrac{1}{12}.$$

Hence force required to raise 112 lb. = 112/12 = $9\frac{1}{3}$ lb. weight.

EXAMPLES XXII.

NOTE.—The student is at liberty to apply the Principle of Work or Principle of Conservation of Energy to any problem whatever in Mechanics, provided its use is not precluded by the conditions of the question (as, for example, where it is required to *verify* the principle so that its truth must not be *assumed*).

1. A weight of 35 lb. balances a weight of 15 lb. at the extremities of a uniform lever 15 ft. long. Find the lengths of the arms.

2. The arms of a lever are 8 in. and 12 in. in length, and the weight 6 lb. is attached to the shorter arm. Find the power.

3. If one end of a bar rests on a beam, and a weight of 60 lb. be suspended from it one-fifth of its length from the beam, what power at the other end will support the weight, and what will be the pressure on the beam?

4. When two weights of 12 lb. and 4 lb. are suspended at the ends of a weightless lever, the fulcrum is 9 ft. from the smaller weight. Where must the fulcrum be when the weights are each increased by 4 lb.?

5. The pressure on the fulcrum of a lever of the second class is 7 lb., and the sum of the effort and weight is 13 lb. Find their distances from the fulcrum when they are 14 in. apart.

6. A lever of the second class, 5 ft. long, supports a weight of 63 lb., and the pressure on the fulcrum is $3\frac{1}{2}$ times the power. Find the power and the position of the weight.

7. A lever of the third class has the power $5\frac{1}{2}$ times the weight, and the pressure on the fulcrum is 27 lb. Find the weight.

8. A man who weighs 160 lb., wishing to raise a rock, leans with his whole weight on one end of a horizontal crowbar 5 ft. long, which is propped at a distance of 4 in. from the end in contact with the rock. What force does he exert on the rock, and what pressure has the prop to sustain?

9. A straight lever, 6 ft. long and heavier towards one end, is found to balance on a fulcrum 2 ft. from the heavier end; but, when placed on a fulcrum at the middle, it requires a weight of 3 lb. hung at the lighter end to keep it horizontal. Find the weight of the lever.

10. A weight of 56 lb. is attached to a straight lever without weight at a distance of 3 in. from the fulcrum, and is balanced in one case by a power of 6 lb., and in another case by a power of 16 lb. Find, in each case, the pressure on the fulcrum, and also the distance between the points of application of the power and the weight when they are applied (i) on the same side of the fulcrum, (ii) on opposite sides of the fulcrum.

11. Two weights P and Q balance on a weightless lever, the fulcrum being $1\frac{1}{4}$ in. from the middle point of the lever. If each weight be increased by 1 lb., the fulcrum must be moved $\frac{1}{4}$ in. in order to have equilibrium. Find the pressure on the fulcrum in the first case.

12. A lever is in equilibrium under the action of the forces P and Q, and is also in equilibrium with the same fulcrum where P is doubled, and Q is increased by 5 lb. Find the magnitude of Q.

13. Two forces of 4 lb. and 8 lb. act at the same point of a straight lever, on opposite sides of it, and maintain equilibrium, the less force being perpendicular to the lever. Find the direction of the greater force.

14. The pressure on the fulcrum of a straight lever is 12 lb., and the difference of the forces is 4 lb. Find the forces and the ratio of the arms at which they act.

15. Two weights hang from the ends of a weightless lever bent at right angles, with the angle for fulcrum, and having one arm 3 times as long as the other. If, in the position of equilibrium, the longer arm makes an angle of 30° with the vertical, find the ratio of the two weights.

16. A weightless lever is 5 ft. long, and from its ends a weight is supported by two strings 3 ft. and 4 ft. long respectively. Find the ratio of the lengths of the arms when the lever is horizontal.

17. If the pressure on the fulcrum be 10 lb., and one of the weights be distant from the fulcrum one-fifth of the whole length of the lever, find the weights, supposing them on opposite sides of the fulcrum.

18. With a wheel and axle, a power of 3 lb sustains a weight of 48 lb., and the radius of the wheel is 2 ft. Find the radius of the axle.

19. Find the power which will support a weight of $1\frac{1}{2}$ tons, if the circumference of the wheel and the axle are respectively 56 in. and 6 in.

20. The drum of a windlass is 4 in. in diameter, and the power is applied to the handle 20 in. from the axis. Find the force necessary to sustain the weight of 100 lb., and the work done in turning the handle 10 times.

21. Find what weight suspended from the axle can be supported by a weight of 10 lb. suspended from the wheel, the radii of the wheel and axle being 2 ft. and 4 in. respectively.

22. The difference of the radii of a wheel and axle is 35 in., and the weight is 8 times the power. Find the radii of the wheel and axle.

23. A wheel and axle is used to raise a bucket weighing 36 lb. from a well. The radius of the wheel is 24 in., and while it makes 7 revolutions the bucket rises 11 ft. Find, by the Principle of Work, the force which will just raise the bucket.

24. A capstan, turned by 10 men, is used to haul up an anchor; the levers, at the ends of which the men push, are 12 ft. long, and the radius of the axle is 18 in. When each man is pushing with a force of $1\frac{1}{2}$ cwt., they can just raise the anchor. Find the weight of the anchor.

25. Draw to scale a wheel and axle, by which a man, sitting in a loop at the end of a rope wound round the axle, can haul himself up by pulling at a rope round the wheel with a force only one-fifth of his weight. Find the weight sustained by the pivots.

26. A uniform straight rod without weight is bent at its middle point so as to form an angle of 105°. It is supported at the angle, and it is found that, when weights P and Q are suspended from the ends, the arm to which the weight P is attached makes an angle of 60° with the vertical. Show that $3P^2 = 2Q^2$.

*27. In a weightless straight lever of the first class, show that in the case of equilibrium the power, the weight, and the reaction of the fulcrum form two unlike couples of equal moment.

CHAPTER XXIII.

MACHINES: THE PULLEYS AND SCREW.

249. The pulley, or pully,* is a wheel with a *groove* cut round its rim so that it can carry a string or rope or chain passing round it. It turns on an axis or axle, which is fixed in a framework called a *block* or *sheave*, and this block is either fixed, or is attached to a string and is then moveable.

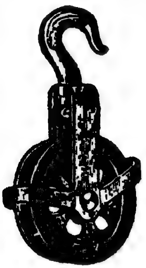


Fig. 124.

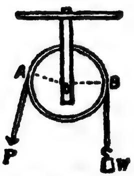


Fig. 125.

250. In the fixed pulley the weight is attached to one end of the string passing round the groove, and the effort is applied by pulling the other end. Since the wheel is only supported by the axle, the moments of the effort P and weight W about the centre O are equal and opposite; that is (Fig. 125),

$$P \times OA = W \times OB,$$

or
$$P \times \text{radius of pulley} = W \times \text{radius of pulley}.$$

$$\therefore \quad P = W,$$

$$\therefore \quad \text{mechanical advantage } \frac{W}{P} = 1 \quad \dots\dots\dots\dots\dots(1).$$

* The word may be spelt either *pulley*, plural *pulleys*, or *pully*, plural *pullies*.

Thus exactly the same force must be applied to lift a given weight as without the pulley.

COR. Since $P = W$, the tension is the same in both parts of the string.

251. In discussing the equilibrium of forces acting on pulleys, we may therefore assume that *the tension is the same throughout every part of the same string, no matter how many pulleys it may pass round.* We likewise suppose the strings to be weightless, and perfectly flexible (and also inextensible).

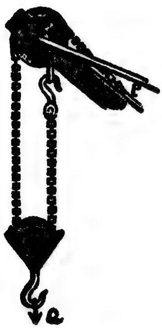


Fig. 126.

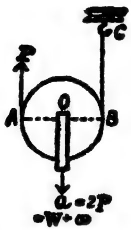


Fig. 127.

252. In the single moveable pulley, the weight is attached to the block, and the effort is applied to one end of the string which passes round the pulley, the other end being fixed up.

In this arrangement, if the strings are parallel, the mechanical advantage is 2.

For let P be the effort, Q the total weight to be raised (*including the weight W of the pulley itself*). Then the tension in each part of the string is equal to P (§ 251); also the tensions at A, B (Fig. 127) both help to support the weight Q.

$$\therefore\ Q = 2P,$$

and mechanical advantage $Q \div P = 2$............(2).

253. If w is the weight of the pulley itself, W the weight of the attached load, $Q = W + w$; and, therefore,

$$W + w = 2P \; \ldots\ldots \; \text{(2a)}.$$

The single moveable pulley is much used on cranes (Fig. 126).

254. The Single-String System of Pulleys.*—A greater mechanical advantage may be obtained with a number of pulleys. Several such "systems of pulleys" are generally described, but the most practically useful system is that in which the pulleys are arranged in two blocks, one fixed and the other attached to the weight (Fig. 128). The same string passes round all the pulleys; it passes alternately over a fixed and under a moveable pulley, and is finally attached to one of the blocks.

In practice the pulleys are arranged as in Fig. 128, but it is generally easier to draw the diagram as in Fig. 129, with all the pulleys in the same plane.

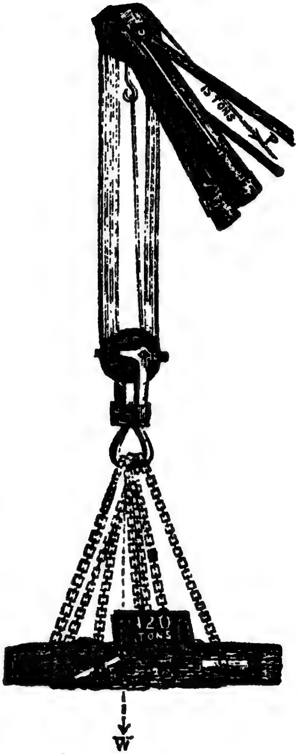


Fig. 128.

255. Mechanical Advantage.—If the applied effort be a force P, the tension throughout is P. Hence, if n be the number of parts of the string supporting the lower block, Q the weight to be raised (including that of the lower block and pulleys),

* The single-string system is often called the *second system* of pulleys, and the separate-string system of § 258 is then called the *first system*.

the n upward forces P support Q acting downwards; hence, supposing the parts of the string vertical,

$$Q = nP \text{.................} (3).$$

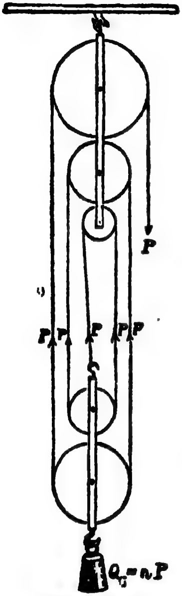


Fig. 129.

$$\therefore \textbf{ mechanical advantage } \frac{Q}{P} = n.$$

256. If w is the weight of the lower block, W that of the attached load, $Q = W + w$; and, therefore,

$$W + w = nP \text{............} (3a).$$

In order that the effort may be applied *downwards*, the free end of the string must hang from a fixed pulley, and this is almost invariably done for convenience in working the system. In such cases the number n is also the total number of pulleys in the two blocks.

Thus Fig. 128 represents a system with altogether 8 pulleys, in which the mechanical advantage is therefore 8. In Fig. 129, there are altogether 5 pulleys, and the mechanical advantage is 5.

257. We would here call the student's attention to the twofold aspect of a *tension*, or *stress* of any kind.

Referring to Fig. 129, the tension in the string is P. The upward arrows indicate the upward forces on the lower block, but there are equal downward pulls on the upper block, the total pull on which is easily seen to be $(n + 1)P$. So, too, in Fig. 127, the string exerts upward pulls P at A, B, and a downward pull P at C.

Example.—To find the least number of pulleys in a moveable block weighing 10 lb., in order that a weight of 120 lb. may be lifted by a downward force not exceeding 28 lb., and to find this force.

Let n be the total number of portions of the string supporting the lower block, P the required effort. Then the pulls P in the strings

have to support both the attached weight of 120 lb. and the block weighing 10 lb.; therefore

$$nP = 120 + 10 = 130 \text{ lb.}$$

But P is *not more* than 28 lb.; therefore n must be greater than $4\frac{9}{14}$; that is, $n = 5$.

Hence five parts of the string must support the lower block. Therefore that block must contain two pulleys, and must have the end of the string attached to it as well (Fig. 129). Also, putting $n = 5$, we have $5P = 130$ lb.; $\therefore$ required force $P = 26$ lb.

258. **The separate-string system of pulleys*** consists of a number of single moveable pulleys like that described in § 252, so arranged that the string hanging from one pulley passes round the pulley next below, the other ends of the strings being attached to a fixed beam or other support (such as the mast of a ship), considerably above the highest points to which weights have to be raised (Fig. 130).

259. **The mechanical advantage** may be found thus:—

In the single moveable pulley a force P applied to the string supports a force $2P$ applied to the block.

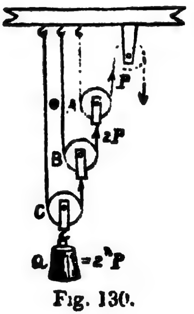


Fig. 130.

Now suppose the moveable pulley, instead of being attached to the weight, supports a string passing under a second moveable pulley. Then the mechanical advantage gained by the first pulley is evidently doubled by the second, the pull $2P$ in the second string supporting a weight $4P$ attached to the second pulley.

Next suppose the second pulley supports a string passing under a third pulley. This again doubles the mechanical advantage, and the system will now support a weight $8P$.

In this way each additional pulley doubles the mechanical advantage of the system. By using 1, 2, 3 pulleys, we get mechanical advantages 2, 4, 8

* The so-called *first system*.

Generally, let there be n pulleys, and let Q denote the weight attached to the last pulley. Then, if we leave out of account the weights of the pulleys themselves, we have

$$Q = 2^n P.$$

Therefore also $\quad P = \dfrac{Q}{2^n}$

and $\quad$ mechanical advantage $= 2^n$ $\Big\}$ ....(4).

260. If the weights of the pulleys **A**, **B**, **C** are w_1, w_2, w_3 respectively, we must consider the equilibrium of each pulley separately. If T_1, T_2 be the tensions of the strings hanging from **A**, **B**, and W the load attached to **C**, we have, by § 253,

$$2P = T_1 + w_1, \quad 2T_1 = T_2 + w_2, \quad 2T_2 = W + w_3.$$

From equations such as these, it may be deduced that, for n pulleys,

$$2^n P = W + w_n + 2w_{n-1} + 4w_{n-2} + \dots + 2^{n-1}w_1 \text{ ....(4}a\text{).}$$

Examples.—(1) If there are ($n =$) 4 moveable pulleys, a force of ($P =$) 10 lb. will support a weight

$$(Q =)\ 2^4 P = 2^4 \times 10 \text{ lb.} = 160 \text{ lb.}$$

(2) If there are ($n =$) 3 moveable pulleys, the force required to support a weight ($Q =$) 64 lb. is

$$(P =)\ Q \div 2^n = 64 \div 2^3 = 64 \div 8 = 8 \text{ lb.}$$

(3) What load can be supported by a force of 10 lb. in a system of 3 moveable pulleys whose weights, beginning with the highest, are 1, 2, 3 lb. respectively?

Consider, first, the equilibrium of the highest pulley (**A**, Fig. 130). The forces on it are the two equal pulls of 10 lb. in the two parts of the string round it, acting upwards, and the weight of the pulley (1 lb.) and the tension T_1 of the string next below, acting downwards.

Hence $\quad 2P = 1 + T_1$ ....(1).

Consider, now, the pulley **B**, acted upon by $2T_1$ upwards, and by T_2 and its own weight (2 lb.) downwards.

Hence $\quad 2T_1 = 2 + T_2$ ....(2).

Similarly, for the pulley **C**, we get the equation of forces

$2T_2 = 3 + W$ ....(3).

Since $P = 10$, we find, from equation (1), that $T_1 = 20 - 1 = 19$; then, from equation (2), $T_2 = 2T_1 - 2 = 38 - 2 = 36$; and lastly, from equation (3), $W = 2T_2 - 3 = 72 - 3 = 69$ lb.

(4) What force is required to support a load of 13 lb. in a system of 4 moveable pulleys whose weights, commencing with the highest, are 3, 5, 7, 9 lb. respectively?

Here we are given the weight, and have to find the effort. Hence the process is the reverse of that of Ex. (3). We must begin with the lowest pulley. Adding its weight (9 lb.) to the attached weight (13 lb.), we have the total weight supported by the two pulls (T_3) in the parts of the string round the lowest pulley.

Hence $\qquad 2T_3 = 22$ lb., $\quad T_3 = 11$ lb.

Similarly, in the lowest pulley but one, we have 11 + 7 lb. supported by $2T_2$.

Hence $\qquad 2T_2 = 18$ lb., $\quad T_2 = 9$ lb.

In like manner, $\qquad 2T_1 = 9 + 5, \quad T_1 = 7$ lb.,

$\qquad 2P = 7 + 3, \quad P = 5$ lb.

That is, the required force = 5 lb.

*261. The **inverted separate-string system of pulleys,*** in which the strings are all attached to the weight, is merely the system last described (*i.e.* the "first") turned upside down (the fixed pulley being omitted). The strings are all attached to the weight, or rather to a rod carrying the weight, and the uppermost pulley is fixed to some support. If Fig. 131 be turned upside down, it will present a similar appearance to Fig. 130.

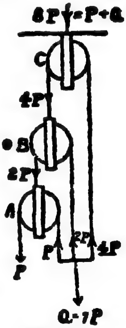


Fig. 131.

*262. The **mechanical advantage** (when the weights of the individual pulleys are neglected) is easily deduced from this property. Let R be the pull which the system exerts on its support, Q the weight, P the effort, and let there be n pulleys. By inverting the system or otherwise, we see that the total forces supported by the several pulleys, commencing with the lowest, are $2P$, $4P$, $8P$, etc.; thus R in this present system corresponds to the weight in the last system, and therefore

$$R = 2^n P.$$

* The so-called *third system*. This system is practically useless (see § 263).

Now consider the equilibrium of the whole system, consisting of the weight and the pulleys. The forces acting on it are Q and P pulling downwards *and a reaction equal and opposite to R holding the system up.*

Hence, $$R = P + Q.$$

$$\therefore \quad Q = R - P = 2^n P - P = (2^n - 1) P,$$

and $$\text{mechanical advantage } Q \div P = 2^n - 1 \quad \ldots\ldots\ldots(5)$$

Example.—To find the number of weightless pulleys, having given that a force of 5 lb. supports a weight of 75 lb., and all the strings being attached to the weight.

Here the pull on the beam supporting the upper pulley

$$= 75 \text{ lb.} + 5 \text{ lb.} = 80 \text{ lb.}$$

Now $80 = 5\times(2\times2\times2\times2)$, and, since the total pull is doubled by each pulley, the number of pulleys must be 4.

*263. *If the weights of the pulleys are taken into account,* we proceed as follows :—

Example.—To find the force required to support a weight of 112 lb. in a system of three pulleys, the weights of the lowest and next pulleys being 3 and 5 lb.,* and the strings all being attached to the weight.

Let P be the force. Then, by considering the equilibrium of each pulley, we have

$$\begin{aligned} &\text{Pull of lowest string} &&= P \\ &\text{Pull in second lowest string } T_1 &&= 2P + 3 \\ &\text{Pull in last string } T_2 = 2T_1 + 5 &&= 4P + 11 \end{aligned}$$

$$\therefore \quad \text{weight supported} = P + T_1 + T_2 = 7P + 14$$

$$\therefore \quad 7P + 14 = 112.$$

$$\therefore \quad \text{required force } P = 14 \text{ lb.}$$

The inverted separate-string system, or "third system," is of no practical use whatever, for it is found that the strings are almost certain to get hopelessly entangled, even in the few working models that are constructed for the lecture-room.

* Since the upper pulley is *fixed,* its weight does not enter into the relation between P and W.

264. Man Raising Himself with a System of Pulleys.—When a man, sitting in a loop or seat suspended by any arrangement of pulleys, pulls *himself* up, the rope which he pulls will support part of his weight, and only the remaining part of his weight will have to be supported by the system.

Examples.—(1) Consider a man of weight W raising himself by pulling a rope passing over a *fixed* pulley with a force P. The man's weight is really supported by the pulls P in *two* parts of the rope—the part where he pulls and the part supporting the loop. Hence the condition of equilibrium gives $W = 2P$;

$$\therefore \quad P = \tfrac{1}{2}W;$$

or the man pulls with a force of *half* his weight.

(2) If the man pulls a rope which passes over a fixed pulley and under a moveable pulley supporting the loop, there are three portions of the string supporting his weight, inclusive of the one that he is pulling. Hence the relation between the pulling force P and the weight is $W = 3P$, or $P = \tfrac{1}{3}W$; so that the man pulls with a force of *one-third* of his weight.

*265. **Applications of the Principle of Work.**—We may also apply the Principle of Work to find the relations between the effort and weight in the various systems of pulleys.

The Single Movable Pulley.—Let the pulley (weight w) and its attached weight W be raised through a height h. Then the portions of string on the two sides of the pulley will each be shortened by h; hence the end at which P is applied will rise through a distance $2h$. Since

sum of works done against W, w = work done by P;

$$\therefore \quad Wh + wh = P \times 2h,$$

or
$$W + w = 2P \qquad \text{.........................(2a)}$$

the required relation between W and P. This agrees with § 253.

*266 **The Single-String System.**—Let there be n strings from which the lower block of pulleys hangs. If the effort P lifts Q through a height h, each of the n portions of string will have shortened by an amount h, and so P will have to pull the end

of the string down through a distance nh. By the Principle of Work, therefore

$$Q.h = P.nh.$$

$$\therefore \quad Q = nP \quad \text{(3a)},$$

the required relation between Q and P, which agrees with § 255.

***267. The Separate-String System.**—Neglecting the weights of the pulleys, and referring to Fig. 130, if Q is drawn up through a height h, the pulley **C** is also raised through h, and, as in § 265, the pulley **B** will be raised $2h$, and **A** will be raised $4h$, and, finally, P must move through a distance $8h$.

Therefore, by the Principle of Work,

$$P \times 8h = Wh, \quad \text{or} \quad W = 8P,$$

which agrees with (4) on putting $n = 3$.

268. The Screw.—Every one is familiar with a **screw.** It consists essentially of a cylindrical bolt **OM**, whose surface carries a *thread* or has a *groove* cut in it along a spiral curve. The form of this spiral can easily be constructed by taking a strip of paper with a straight edge, and wrapping it round a pencil in a slanting direction; the edge forms the curve like that along which the thread or groove runs.

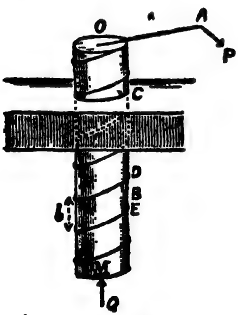


Fig. 132.

The screw works in a *collar* or *nut* **C**, through which a hole is bored, having a groove to fit the thread or a thread to fit the groove of the screw.

When the screw turns in a fixed collar, it moves forward in the direction of its length. In each turn of the screw, the distance moved forward is equal to the distance between consecutive threads: *i.e.* the distance **DE** between two consecutive turns of the thread, measured along the length of the bolt. This distance is called the **step.** Hence,

by turning the screw round, it may be used to raise weights or overcome resistances applied to its end.

The *effort* must tend to turn the screw, and must therefore have a *moment* about **OM** in a plane perpendicular to **OM**. Hence the effort may be a single force P applied at the end of a long arm **OA**, projecting at right angles to **OM**. More often the arm projects in both directions, as in the common screw-press of Fig. 133, and two equal and opposite forces constituting a *couple* are then applied perpendicularly to its two extremities **A**, **B**.

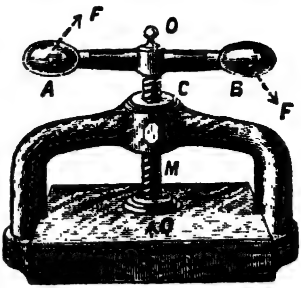


Fig. 133.

269. To find the mechanical advantage of a screw working without friction.

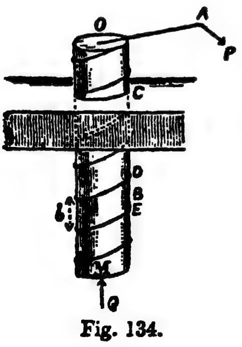


Fig. 134.

Let the effort P be applied perpendicularly at the end of an arm **OA** of length a, and let it overcome a resistance Q acting along the axis **OM**. Let b be the "step" or distance between two consecutive threads. Then, if the screw makes one complete turn, **A** the point of application of P will describe a circle of radius a about **O**, and will therefore move through a distance $2\pi a$ in the direction of P. Also the screw will move through a distance b against the resistance Q.

Therefore, by the Principle of Work, we have $P\times 2\pi a = Q\times b$.

$$\therefore \text{ mechanical advantage } \frac{Q}{P} = \frac{2\pi a}{b} \qquad (6)$$

$$= \frac{\text{circumference of circle described by the arm}}{\text{step of screw}}.$$

Example.—A screw-press is turned by applying two forces of 21 lb. in opposite directions at the ends of an arm 2 ft. long. If the step is $\frac{1}{8}$ in., to find the resistance overcome.

Let Q be the resistance. Then in one revolution of the screw the works done by the efforts and against Q are

$$2\times21\times2\pi.1 \text{ ft.-lb.} \quad \text{and} \quad Q\times\tfrac{1}{8}\times\tfrac{1}{12}\text{ ft.-lb.}$$

$$\therefore \quad \tfrac{1}{96}Q = 84\pi, \quad \text{or} \quad Q = 96\times84\pi.$$

Taking $\pi = \frac{22}{7}$, this gives $Q = 25344$ lb. $= 11\frac{1}{3}$ tons, roughly.

269a. Disadvantage of the Wheel and Axle.—In the simple wheel and axle the "mechanical advantage"

$$\frac{\text{radius of wheel}}{\text{radius of axle}}$$

can be made larger only by

(1) increasing the radius of the wheel; or
(2) diminishing the radius of the axle.

The former leads to an unwieldy machine; the latter arrangement is limited owing to the necessity for the axle being sufficiently strong to bear the forces to which it is subjected. This disadvantage is absent in the modified form known as the *differential wheel and axle.*

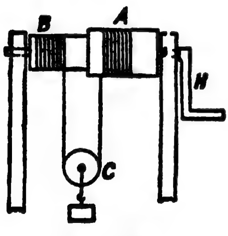


Fig. 134*a*.

269b. The Differential Wheel and Axle, or Chinese Windlass.—In this machine a handle H takes the place of the "wheel," while the "axle" consists of two cylinders (A, B) on a common axis but with different radii. The body to be raised is attached to a moveable pulley C. This is supported by a cord one end of which is wound round A and the other end is wound *in the opposite* direction round B. When the handle H is turned, the cord winds on to A and winds off B.

Let a and b be the radii of the larger and smaller portions of the axle, c the radius of the arm at which the power acts. Let P lb. be the force exerted, W lb. the weight of the body raised.

Then, supposing the strings on both sides of the pulley vertical, when there is equilibrium the tension of the string is $\frac{1}{2}W$,

Taking moments round the axis of the axle, we have, since the vertical portions of the string tend to turn the axle in opposite directions,

$$Pc = \tfrac{1}{2}W.a - \tfrac{1}{2}W.b\,;$$

therefore the mechanical advantage

$$= \frac{W}{P} = \frac{2c}{a-b}.$$

The same result can be obtained by the Principle of Work as follows.

Imagine a complete revolution of the axle to take place. A length $2\pi a$ of string is wrapped on to the larger cylinder, and a length $2\pi b$ is unwrapped from the smaller cylinder. Thus the string not on the cylinders is shortened by $2\pi(a-b)$, and therefore the pulley is raised a distance $\pi(a-b)$.

Therefore the work done against the weight is $W\,\pi(a-b)$.

Also the weight done by the power P in a complete revolution is $P.2\pi c$;

$$\therefore\ P.2\pi c = W\pi(a-b)$$

or the mechanical advantage

$$= \frac{W}{P} = \frac{2c}{a-b}.$$

We see, then, that in this machine the "mechanical advantage" can be made very large merely by diminishing $a-b$; that is, by making the radii of the two cylinders nearly the same. Thus these cylinders can be of medium size, and the mechanical advantage increased as much as desired by making their radii nearly equal.

269c. The Differential Pulley.—The principle employed in the differential wheel and axle is utilised in a more practical machine known as the *differential pulley*. This differs from the pulley of the second system in the following points: The upper pulley, which is the differential portion, consists of

two concentric wheels of different radii, *a and b, cut in one block*. The rope is replaced by a long *endless* chain of the same type as a bicycle chain, and the pulley grooves by cogs on to which the links of the chain fit.

In one complete revolution of the upper pulley the length of chain which passes over the larger circumference of the differential pulley is $2\pi a$, and the length wound over the smaller circumference is $2\pi b$. Hence the total length of chain connecting the upper and lower pulleys is diminished by $2\pi(a - b)$. But this decrease is divided between the two vertical lengths of chain; thus, the weight W is raised a vertical height equal to the half of this decrease, or $\pi(a - b)$

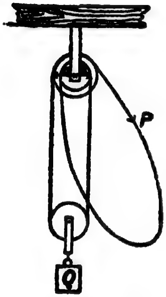


Fig. 134*b*.

Therefore the work done on

$$W \text{ is } W \times \pi(a - b).$$

Also the work done by P is obviously $P \times 2\pi a$.

Equating these two amounts of work we have
$$P \times 2\pi a = W \times \pi(a - b),$$
whence **mechanical advantage**
$$\frac{W}{P} = \frac{2a}{a - b}.$$

The mechanical advantage may be made very great by making b nearly equal to a. In addition to its great mechanical advantage, another useful property of the differential pulley is that when the weight is being raised it can safely be left, as the large friction in the machine will prevent the weight running down.

269d. If a large mechanical advantage were required in a *screw*, it could be secured by making the power-arm very long or by diminishing the "step" considerably. The former would result in a cumbrous machine, while the latter would make the thread too weak for the forces to which it is subjected. These disadvantages are absent in the *differential screw*.

269e.* The Differential Screw.—The screw **M**, which is turned by the lever, works as usual in a *collar* or *nut* in the framework at **C**. To the top of the press **A** is fastened another screw **N** of smaller diameter than **M**. The screw **N** works in a nut which is cut in the lower extremity of the screw **M**, and both **M** and **N** are "right-handed" screws. Also b, the "step" of the screw **M**, is slightly larger than c, the step of the screw **N**.

Now let P be the force applied at the end of the lever, whose length is a, and let Q be the resistance of the press. Then the work done by P in one revolution is $P \times 2\pi a$. Again, in one

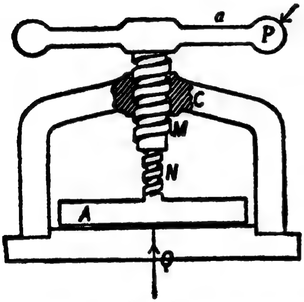


Fig. 134c.

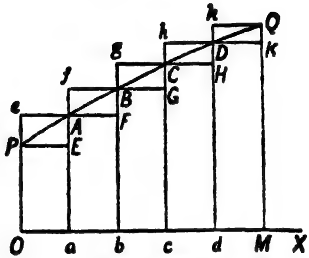


Fig. 134d.

revolution the screw **M** travels downwards through a distance b, while the screw **N** penetrates a distance c into the screw **M**. Hence the press **A** is only forced down through a distance $(b - c)$, and the work done against Q is $Q \times (b - c)$.

Hence from the equation of work we have

$$P \times 2\pi a = Q \times (b - c),$$

$$i.e. \text{ mechanical advantage } \frac{Q}{P} = \frac{2\pi a}{b - c}.$$

Since c is very nearly equal to b, the denominator of this fraction is very small, and the mechanical advantage is very great.

269f. On the Work done by a Variable Force.—The problem of determining the work done by a variable force is similar to that of determining the distance travelled under a variable

velocity; and the method of graphic representation, which we applied to the latter problem in § 41, may with advantage be applied to the former also.

In Fig. 134*d* let the line or curve **PQ** be so drawn that, for any point **B**, the ordinate *b***B** represents the *magnitude of the force* when its point of application has travelled a *distance* represented by the abscissa **O***b*. Then the area **OPQM** (between the curve, the extreme ordinates, and the axis **OX**) will represent the work done by the force.

Work is the product of force and distance; as the point of application moves over a distance represented by *bc*, the magnitude of the force increases from *b***B** to *c***C**. Thus the work done must be greater than would be done by a force *b***B** acting through a distance *bc*, and less than would be done by a force *c***C** acting through a distance *bc*. Hence the work done must be greater than *b***B** × *bc* and less than *c***C** × *bc*, *i.e.* greater than that represented by the area *b***BG***c*, and less than that represented by the area *bg***C***c*. Applying the same argument to each of the distances represented by **O***a*, *ab*, etc., it follows that the total work done is less than that represented by the circumscribed figure **O***e***A***f***B***g***C***h***D***k***QM** and greater than that represented by the inscribed figure **OPEAFBGCHDKM**.

In a similar way we can show that, into however many parts we divide the base line **OM**, the work done by the variable force must be represented by an area which is intermediate in magnitude between the corresponding circumscribed and inscribed figures.

But if the divisions of **OM** become very numerous and very small both the circumscribed and inscribed figures approximate very closely to the area **OPQM**. Hence the area **OPQM** represents the work done by the force.

NOTE.—If the same curve were traced from **Q** to **P** (instead of from **P** to **Q**) this would represent the case where the point of application is travelling in the opposite direction to the force. The area **OPQM** would as before represent the work done, but it would have to be reckoned as negative work.

269g. On the Efficiency of a Machine.—In § 246 we see that, if a machine is frictionless and its moving parts are weightless,

then the work done by the machine against the resistance is equal to the work done on the machine by the effort.

In actual practice the work done by the machine is always less than the work done on the machine, and the ratio

$$\frac{\text{work done by machine against resistance}}{\text{work done on machine by effort}}$$

is called the efficiency of the machine.

Thus the efficiency of a locomotive is the ratio of the work done by the locomotive in pulling the train to the work done on the pistons by the steam.

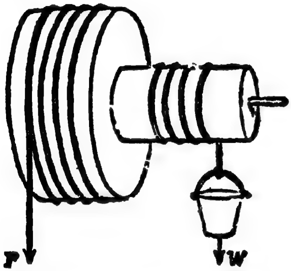


Fig. 134e.

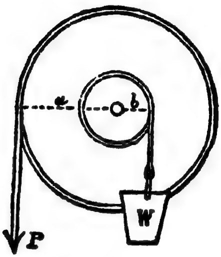


Fig. 134f.

269h. Velocity Ratio.—If we consider any typical weightless and frictionless machine (say for example the wheel and axle, Figs. 134e, 134f) we have an effort (P) acting in the machine at one point, and the machine overcoming a resistance (W) at another point.

DEFINITION.—*The* ***velocity ratio*** *of a machine is the ratio of the velocity of the effort to the velocity of the resistance.*

I.e. in this case the ratio of the velocity of P to that of W.

We can now show that *the mechanical advantage of such a machine is equal to its velocity ratio.*

Suppose that in Fig. 134e the velocity of P is u ft./sec. and that of W is v ft./sec.

Then in one second P moves a distance u downwards, and W moves a distance v upwards. By the Principle of Work—

$$\text{work done by } P = \text{work done by } W,$$

hence
$$P \times u = W \times v.$$

$$\therefore \frac{W}{P} = \frac{u}{v}, \text{ i.e. } \textit{mechanical advantage} = \textit{velocity ratio.}$$

Example.—Calculate the velocity ratio of the wheel and axle without using the Principle of Work; deduce the mechanical advantage.

Using figure 134*f*—In one revolution of the machine P describes a downward distance equal to the circumference of a circle of radius a, and W describes an upward distance equal to the circumference of a circle of radius b.

Thus $$\textit{velocity ratio} = \frac{\text{vel. of } P}{\text{vel. of } W} = \frac{2\pi a}{2\pi b} = \frac{a}{b}.$$

Also *mechanical advantage* = velocity ratio = $\frac{a}{b}$.

The above assumes no loss of energy in the machine. In practice there is always some loss and the mechanical advantage depends on the energy wasted. In such cases the relation is:—mechanical advantage = velocity ratio × efficiency: thus:—

$$\text{Efficiency} = \frac{\text{useful output of work}}{\text{input of work}} = \frac{\text{work done against resistance}}{\text{work done by effort}}$$

$$= \frac{\text{resistance} \times \text{distance moved by resistance}}{\text{effort} \times \text{distance moved by effort}} \textit{ in the same time}$$

$$= \frac{\text{mechanical advantage}}{\text{velocity ratio}};$$

$\therefore$ mechanical advantage = velocity ratio × efficiency.

If the efficiency be unity (100 per cent.) the mechanical advantage is equal to the velocity ratio as previously indicated.

EXAMPLES XXIII.

1. In a single moveable pulley with parallel strings find the force required to support a weight of 1 cwt., the weight of the pulley being 8 lb.

2. A man, whose weight is 180 lb., is seated in a loop at one end of a rope which passes over a smooth fixed pulley, and he grasps the other end of the rope with both hands. Find the weight supported by each

of his arms, assuming that they support equal weights and that the two portions of the rope are parallel.

3. Find the ratio of the power to the weight in that system of pulleys in which each moveable pulley hangs by a separate string, the number of moveable pulleys being 4. (First system.)

4. If the weights of the pulleys in the preceding Example be taken into account, and be 1, 2, 3, and 4 lb. respectively, beginning with the lowest, what weight will be supported by a power of 5 lb.?

5. If in the first system with weightless pulleys a power of 1 oz. supports a weight of 8 lb., find the number of moveable pulleys.

*6. Find the ratio of the power to the weight in that system of pulleys in which each string is attached to the weight, the number of moveable pulleys being three.

*7. If the weights of the pulleys in the preceding Example be taken into account, and be 1, 2, and 3 lb. respectively, beginning with the lowest, find the power requisite to support a weight of 31 lb.

8. If, in the system of pulleys in which each moveable pulley hangs by a separate string, a man supports a weight (W) equal to his own, and there are 5 moveable pulleys, find his pressure on the floor on which he stands.

9. Is it more advantageous in raising a weight by (i) the first system, (ii) the third system, of pulleys to have the pulleys heavy or light?

*10. If, in a system of pulleys in which each string is attached to the weight, there be three moveable pulleys, each weighing 1 lb., and the power required to support a certain weight is half that which would be required if the pulleys were weightless, find that weight.

11. In the single-string system of pulleys find the number of strings at the lower block if a power of 8 oz. supports a weight of 8 lb.

12. Find the weight which can be supported by a power P used in connection with a block and tackle, the number of pulleys in the lower block, which weighs $2P$, being 8, and the string being fastened to the upper block.

13. In the first system of pulleys, with three moveable pulleys, find the weight which can be supported by a power of 80 lb., the weights of the pulleys, beginning with the lowest, being 7, 5, and 3 lb.

*14. If, in the system of pulleys in which each string is attached to the weight, there be four moveable pulleys whose weights, beginning with the lowest, are 2, 3, 4, and 5 lb. respectively, find the power which will support a weight of 440 lb.

15. In the second system of pulleys, there are four pulleys in the lower block, which weighs 24 lb., and the string is fastened to the lower block. Find the weight which can be raised by a power of 16 lb.

16. If there are six parts of the string at the lower block of a block and tackle, find the greatest weight which a man weighing 12 stone can possibly support.

17. If a man weighing 10 stone supports a weight of 5 stone by means of a block and tackle having three pulleys in the lower block and the rope attached to that block, find the pressure on the floor on which he stands, and also the pressure on the beam to which the upper block is fastened.

18. If, in the first system, the weight of each moveable pulley be $\frac{1}{16}P$, find the weight that will be supported by a power P when there are four moveable pulleys.

19. In the last question find the pressure on the beam to which the strings are fastened.

20. If there are five moveable pulleys in the third system, each weighing 1 lb., show that a power of 4 lb. will support a weight of 309 lb.

21. There are four pulleys in the lower block of a block and tackle, and the rope is attached to the upper block. It is required to raise a box 4 ft. How much rope must be used?

22. How would you arrange a number of weightless pulleys so that a force equal to the weight of 1 lb. would just support a weight of 63 lb.?

23. Describe and sketch a system of pulleys on which (neglecting the weight of the pulleys) a power of $10\frac{1}{2}$ lb. would balance a weight of 84 lb., and show how far the power must move in order to raise the weight 3 ft.

24. If there be four moveable pulleys in the first system, and each weighs 2 lb., what weight can be raised by a force of 20 lb.?

25. Find the ratio of the power and the weight in a screw which has 10 threads to the inch and is moved by a power acting perpendicular to an arm a foot long.

26. A screw whose step is $\frac{1}{4}$ inch is turned by means of a lever 4 ft. long. Find the power which will raise 55 cwt.

27. The arm of a screw-jack is 1 yard long, and the screw has two threads to the inch. What force must be applied to the arm to raise a weight of 1 ton?

28. Find the work done when a force equal to the weight of 7 lb. revolves 5 times tangentially round a circle of 10 ft. radius.

29. The diameter of a screw is 7 in., and the distance between the threads is $\frac{1}{2}$ in. What power must be applied at the circumference of the screw in order to support a weight of 440 lb.?

30. In an ordinary screw-press the power is applied at the ends of two levers each 2 ft. long, and the step of the screw is $\frac{1}{10}$ in. Find the force which must be applied to each lever to produce a total pressure of 2 tons.

*31. In a screw-press a power equal to 10 lb. weight, acting on an arm 3½ ft. long produces a pressure of 4 tons. What is the "step" of the screw?

32. Find the force necessary to raise a weight of 2 tons by means of a differential wheel and axle, if the diameter of the wheel is 6 ft. and those of the portions of the axle are 5 in. and 7 in.

33. In a differential wheel and axle the radii of the two parts of the axle are 5 in. and 4½ in. respectively. The force is applied at the end of an arm 16 in. long. Find the ratio of the velocity of the end of the arm to that of the weight raised.

34. Prove, by the principle of moments, the relation between the power and the weight in the differential pulley.

35. In a differential pulley the diameters of the pulleys in the upper block are 6 in. and 5½ in. respectively. Find the weight that can be supported by a force of 10 lb.

36. A block of stone is to be raised by a differential pulley to a height of 10 ft. The radii of the two wheels are 8¾ in. and 7¾ in. Through how many feet must the power be exerted?

37. The "steps" in a differential screw are ¼ and ⅕ in. Find the pressure that can be exerted by means of it if a power of 3 lb. be applied at the end of an arm 8 in. long.

38. Determine the mechanical advantage of a differential screw composed of a screw of 5 threads to the inch and a screw of 6 threads to the inch, the power being applied at the circumference of a wheel 4 ft. in diameter.

39. In a single-string system of pulleys there are 4 pulleys on the lower block, and the string is fastened to the lower block. If it takes a force of 20 lb. to raise a mass of 110 lb., find the efficiency of the pulley system.

40. A variable force drives a body 32 feet along the ground. If the following table gives the magnitude of the force at different points of the motion, find the total work done by the force.

Distance from the start in feet	0	4	8	14	19	22	23	32
Force in lb. wt.	44·4	35·2	28	20·7	15·8	13·6	11	10·2

41. A crane is lifting a weight of 3 tons at the rate of 1 ft./sec. and the engine is working at the rate of 14 h.p. Find the efficiency of the engine.

42. A waterfall is to be used to drive a turbine, ten tons of water passing through per minute. If the height of the fall is 50 feet find what h.p. can be supplied by the turbine if the efficiency is 45 per cent.?

43. (i) If the velocity ratio of a machine (working with friction) is $\frac{u}{v}$, and the mechanical advantage is $\frac{W}{P}$, find the efficiency.

(ii) Conversely, given that the velocity ratio is $\frac{u}{v}$ and the efficiency is p per cent., find the mechanical advantage.

EXAMINATION PAPER XII.

1. Classify the different kinds of levers, giving examples, and point out those in which there is a mechanical advantage.

Explain clearly what is meant by the statement: "What is gained in power is lost in speed." Show that it is true in the case of a lever of the first class.

2. A uniform weightless lever, 12 ft. long, is in equilibrium when weights of 3 lb. and 9 lb. are suspended from its ends. How far will the fulcrum be shifted when 1 lb. is added to each weight?

3. If a man has to raise a weight, and has only one pulley at his disposal, show how he must employ it in order to obtain the utmost advantage.

4. A weight is to be raised by means of a rope passing round a horizontal cylinder 10 in. in diameter, turned by a winch with an arm 3 ft. 4 in. long. Find the greatest weight which a man could so raise without exerting a force of more than 25 lb. on the handle of the winch.

5. Find the mechanical advantage of n moveable pulleys arranged in the separate-string system; and draw the figure with five pulleys.

6. In the first system of pulleys, what weight will a power of 12 lb. support if there are three moveable pulleys each weighing $\frac{1}{2}$ lb.?

7. State the Principle of Work, and apply it to find the relation between the effort and the weight in the single-string system of pulleys, the strings being parallel and the number of pulleys n.

8. Draw the figure of the third system of pulleys with four moveable pulleys; and prove that, if there are n pulleys including the fixed one, the mechanical advantage in this system is $2^n - 1$.

9. In the second system of pulleys, if a weight of 3 lb. supports a weight of 18 lb., and a weight of 6 lb. supports a weight of 39 lb., find the weight of the lower block. What would be the mechanical advantage if the lower block were weightless?

10. In a screw-press a power of 20 lb., acting on an arm 4 ft. long, produces a force of pressure of 7 tons. What is the step of the screw?

PART III.

CENTRES OF GRAVITY.

CHAPTER XXIV.

CENTRES OF PARALLEL FORCES: DEFINITIONS.

270. We have frequently assumed the fact that the weight of a heavy straight uniform rod or beam may be supposed to act at its middle point, and, generally, that the weight of a rigid body of any shape may be supposed to be concentrated at a single point, called the centre of gravity of the body. We shall now prove this property, and in Chap. XXV. we shall show how to determine the position of this point for bodies of certain shapes.

We commence by proving a few further theorems about parallel forces.

271. To find the resultant of any number of parallel forces of given magnitudes applied at given points of a rigid body, not necessarily in the same plane.

Let the given parallel forces be P acting at **A**, Q at **B**, R at **C**, S at **D**, and so on.

Join **AB**, and in **AB** take a point **E** such that

$$P \times \mathsf{AE} = Q \times \mathsf{EB}.$$

Then, by § 210, the forces P and Q are equivalent to a single resultant force $P+Q$ acting at **E**, parallel to both of them.

Now compound this resultant with the force R at **C**. Join **EC**, and on it take a point **F**, such that

$$(P+Q) \times \mathsf{EF} = R \times \mathsf{FC};$$

then the forces $P+Q$ at E and R at C are equivalent to a single resultant force $P+Q+R$ acting at F parallel to them.

This result is therefore the resultant of the *three* forces P, Q, R at A, B, C, respectively.

Now join FD, and in it take a point G, such that

$$(P+Q+R)\times \text{FG} = S\times \text{GD};$$

then the forces $P+Q+R$ at F and S at D are equivalent to a single resultant $P+Q+R+S$ acting at G parallel to them.

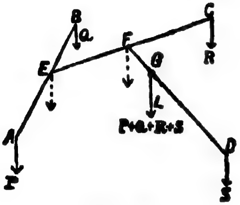


Fig. 135.

This resultant is therefore the resultant of the *four* forces P, Q, R, S, at A, B, C, D, respectively.

Proceeding in this way, we may find the resultant of any number of parallel forces acting at different points of a body.

We might have compounded the four forces P, Q, R, S, in any other order whatever. Thus, if we had first found the resultant of R and S, then compounded this with Q, and compounded the resultant so obtained with P, we should have a different construction for the resultant of the four forces. The point G finally arrived at will be the same in every case, as will be shown in § 273 and § 277.

272. The Centre of Parallel Forces: DEFINITION.—The above construction shows that—

The resultant of any number of parallel forces passes through a certain point whose position depends only on the magnitudes and the points of application of the forces and not on their direction.

This point is called the **centre** *of the parallel forces.*

Thus, in Fig. 135, E is the centre of P and Q, F is the centre of P, Q, R, and G is the centre of P, Q, R, S.

If the forces P, Q, R, S, instead of acting as in Fig. 135, are applied at the same points A, B, C, D, but in a different direction (though still parallel), as in Fig. 136, their resultant will still pass through the same point G as before. And this can be said of no other point.

OBSERVATION.—The student should carefully notice the difference between the *point of application of the resultant* and the *centre* of a system of parallel forces.

If P, Q, R, S act at A, B, C, D *in a given direction*, their resultant acts in a straight line GL, *parallel to this direction and passing through the centre* G. The resultant *may be supposed to act at* G, but it *need not necessarily be applied* at G; for, by the Principle of Transmission of Force, it may be applied at any point (say L) in its line of action (Fig. 135). But let P, Q, R, S, still acting at A, B, C, D, be turned round into a different direction. Their resultant will still pass through G, the centre of the forces, but it will no longer pass through L (Fig. 136).

273. A System of Parallel Forces cannot have more than one Centre.—For the resultant of several parallel forces cannot pass through two definite points except when the forces are parallel to the line joining the points.

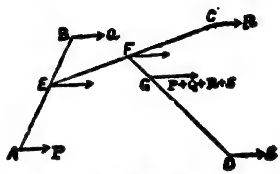


Fig. 136.

274. Centre of Parallel Forces acting at Points in a Straight Line.—If a number of parallel forces P_1, P_2, P_3 ... are applied at points A_1, A_2, A_3 ... along a straight rod or beam, the construction for the centre of parallel forces shows that this point also lies in that straight line. If x_1, x_2, x_3 ... are the distances of A_1, A_2, A_3 ... from a fixed point O on the line, the distance from O of the centre of the parallel forces is therefore given by the formula of § 220,

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \dots}{P_1 + P_2 + P_3 + \dots}.$$

For, since the position of the centre does not depend on the direction of the forces, we may suppose the forces applied perpendicular to the rod, and the result then follows at once by taking moments about O, as in the proof of § 220.

275. Every Body has a Centre of Gravity.—Defining *gravity* as the attraction which the Earth exerts on all bodies, and the *weight* of a body as the force with which that body is attracted to the Earth, it is known that at any given place the weights of different bodies are proportional to their masses.

The weight of a body always tends to pull it towards the centre of the Earth. Hence, defining the *vertical* as the direction of gravity, the verticals at different places would, if produced, meet in the Earth's centre. But the Earth's radius

is nearly 4000 miles; consequently the verticals at two places would have to be produced nearly 4000 miles below the Earth's surface before they would meet. Hence, unless the places themselves are at a considerable distance apart, the verticals are very approximately parallel, and we shall consider them as parallel in treating of the centre of gravity.

Now suppose a body to be built up of a number of heavy particles rigidly connected together. Then the weights of the particles will form a system of parallel forces, and, by § 272, these forces will have a centre through which their resultant always passes. This centre is called the **centre of gravity** of the particles.

The same thing is true for any body, whatever be its nature or the distribution of its parts. For if we subdivide any body into a very large number of parts, we may make these parts so small that each may (for all practical purposes) be regarded as a single heavy particle. Hence the resultant weight due to the weights of the different portions of a body always passes through a certain point in the body, and this point is called the **centre of gravity** of the body.

The centre of a system of parallel forces was defined by the property that the resultant force continues to pass through this point, even when the direction of the forces is changed, provided that they remain parallel forces and have the same magnitudes as before. We cannot alter the direction of gravity, but it will amount to the same thing if, instead, we turn the body itself round, provided that in doing so we do not alter its size and shape, and that we suppose the centre of gravity to move as if rigidly connected with it.

The centre of gravity of a body is therefore *fixed relative to the body*, so that, when the body moves as a rigid body, the centre of gravity moves with it.

The centre of gravity *need not be in the body itself*. Thus the centre of gravity of a circular ring of wire is at the centre of the circle, not in the wire.

It is not necessary that the body itself should be rigid in order to have a centre of gravity, but the centre of gravity remains fixed relative to the body only as long as its size and shape remain unaltered.

Thus a straight piece of wire has a centre of gravity at its middle point, and its weight will act at that point as long as it remains straight. If the wire be bent, it will no longer have the same centre of gravity. A bicycle has a centre of gravity through which its weight will always act, so long as its different parts (the steering-wheel and handle, and the frame, etc.) preserve the same relative position.

276. Definition of the Centre of Gravity.—We may therefore give the following important definition :—

DEFINITION.—*The* **centre of gravity** *of a body is that point, fixed relative to the body, through which the resultant force due to the Earth's attraction on it always passes, whatever be the position of the body, so long as its size and shape remain constant.*

In short, *the centre of gravity is the point at which the whole* **weight** *of the body may always be supposed to act.*

The abbreviation for centre of gravity is **C.G.**

277. **A Body cannot have more than one C.G.**—For if the body had two centres of gravity **G, H,** the line of action of the body's weight would always have to pass through both **G** and **H,** and would therefore have to be the line **GH.** But this is impossible, except when **GH** is vertical, for the weight of a body always acts vertically. Hence the body cannot have two C.G.'s.

[*Compare this proof with* § 273.]

278. Centre of Mass.—Supposing gravity were not to exist, a body would not have *weight*, but it would still have what is called *mass*, although its mass could not be measured in the ordinary way by *weighing*, and the analogy naturally suggests that the body would not strictly have a C.G., but would still have a *centre of mass*.

DEFINITION.—The **centre of mass**, or **mass-centre**, of a body is the centre of a system of parallel forces acting on all the particles into which the body may be supposed to be divided, the force on each particle being proportional to its mass.

The abbreviation for centre of mass is **C.M.**

We observe that every body, or system of bodies, has always a C.M. If subject to gravity, its C.G. will *necessarily*

coincide with its C.M., because the weights of the particles of the body, being proportional to their masses, form a system of forces having the C.M. for their centre.

279. Construction for the C.G. of a Number of Particles.—If particles, whose *weights* are P, Q, R, S, are situated at points A, B, C, D, the point G, obtained by the construction of § 271, will be the centre of parallel forces of magnitudes P, Q, R, S acting at A, B, C, D respectively, and will therefore be the C.G. of the particles.

Similarly for particles in a straight line the C.G. is given by the formula of § 274.

280. Definitions.—A **lamina** is a sheet of material whose thickness is so small that it may be regarded as a distribution of matter over an area. A **uniform** lamina is one which is of the same thickness and formed of the same substance throughout.

The C.G. of an area or surface means the C.G. of a uniform lamina covering that area or surface.

Thus a sheet of paper or thin card, a thin sheet of metal such as that forming a tin canister, a plate of glass of small thickness, a slate, and in some cases even a wooden plank, may be regarded as a lamina. And the C.G. of a parallelogram will be the C.G. of a sheet of paper or any other uniform lamina forming that parallelogram.

A wire, stick, rod, or beam is said to be **uniform** when it is of the same cross section and of the same substance throughout its length.

The C.G. of a line (whether straight or curved) means the C.G. of a thin uniform wire placed along that line.

281. The C.G. of a straight line is at its middle point.

This is obvious, for there is no reason why it should be nearer one end than the other.

Fig. 137

To prove it, let AB be a straight uniform wire, G its C.G. Then G evidently lies somewhere in the line AB. And if the wire be turned round so that the end B is placed at A and the end A is placed at B, the wire lies along the same line as before, and the position of G must

therefore be unaltered. Hence $GA = GB$, or **G** is the middle point of **AB**.

282. To find the C.G. of a parallelogram.

Let **ABCD** be *either* a uniform lamina or a uniform wire in the shape of a parallelogram. Let its diagonals **AC**, **BD** intersect in **G**. Then shall **G** be the required C.G.

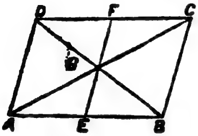


Fig. 138.

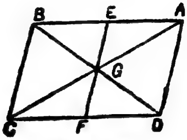


Fig. 139.

Turn the parallelogram round as in Fig. 139, and place it so that the vertices **A, B, C, D** coincide with the previous positions of the vertices **C, D, A, B** respectively. Then the diagonal **AC** will coincide with its previous position **CA**, and **BD** with **DB**. Hence **G**, the intersection of the diagonals, will be unaltered in position. Also no other point in the parallelogram will occupy the same position as before. For any other point will be brought round to the opposite side of **G**.

Now, when the parallelogram is turned round, it occupies the same space as before; hence its C.G. in its new position coincides with its C.G. in its old position. Therefore the C.G. of the parallelogram must be at **G**.

COR. 1. Since the diagonals of a parallelogram bisect each other, *the* C.G. *is at the middle point of either diagonal.*

This is also obvious from the method of proof. When the parallelogram is turned round, **AG** is brought into coincidence with the former position of **GC**. Therefore $AG = GC$, and **G** is the middle point of **AC**.

COR. 2. *The* C.G. *is the middle point of the line bisecting a pair of opposite sides.*

For let **E, F** be the middle points of **AB, CD**. When the parallelogram is turned round, **AB, CD** occupy the former places of **CD, AB** respectively. Hence **E, F** occupy the former places of **F, E** respectively; and therefore the middle point of **EF** occupies the same position as before. Therefore this point is **G**, the required C.G. of the parallelogram.

COR. 3. Similarly, **G** is the middle point of the bisector of the pair of opposite sides **BC, DA**. *Hence the diagonals and the bisectors of the*

opposite sides of the parallelogram all bisect one another in the C.G. *of the parallelogram*, as may be otherwise proved by geometry.

283. **Other Symmetrical Figures.**—From similar considerations of symmetry, we are able at once to write down the following additional results :—

The centre of gravity of

(1) *a circular ring* is the centre of the circle ;
(2) *a circular area* is the centre of the circle ;
(3) *a regular polygon* is the centre of the polygon
(4) *a sphere* is the centre of the sphere ;
(5) *a right cylinder* is the middle point of its axis ;
(6) *a cube* or *rectangular parallelepiped* is at the intersection of its diagonals.

EXAMPLES XXIV.

1. Equal parallel forces act at the angular points A, B, C of a triangle ABC. Find their centre (i) when the forces are like, (ii) when the force acting at C is unlike the other two.

2. A weightless plate in the form of an equilateral triangle is suspended by three parallel strings attached to its angular points A, B, C. The strings can support weights of 1, 1, and 2 lb. respectively. Where must a weight of 4 lb. be placed on the plate so as to be supported?

3. A beam, 10 ft. long and weighing 28 lb., balances about a point 3 ft. from one end. When a weight is hung from the other end, the beam balances about a point $1\frac{2}{3}$ ft. from that end. Find the weight.

4. A uniform bent lever, the weights of whose arms are 6 lb. and 12 lb., rests with its shorter arm horizontal. What weight must be attached to the end of the shorter arm in order that the lever may rest with the long arm horizontal?

*5. Parallel forces of 2, 4, 8, and 10 lb. act at the corners A, B, C, D of a square ABCD. Find the position of their centre.

6. Equal parallel forces act at the six corners of a regular hexagon. Find the position of their centre when one of them is unlike the remaining five.

*7. If a system of parallel forces can be reduced to a couple, what is the position of their centre?

8. Two weights of 4 lb. and 8 lb. balance on a uniform heavy lever whose arms are 5 in. and 3 in. long respectively. Find the weight of the lever.

9. Three spheres, whose weights are 2 lb., 5 lb., and 7 lb., are placed so that their centres are in a straight line, the distance between the centres of the first and second being 14 in., and between the centres of the second and third 10 in. Find their C.G.

10. Weights of 2, 4, 3, 5, and 6 lb. are hung at intervals of a foot along a uniform heavy lever 4 ft. long and weighing 16 lb. Find the position of the fulcrum when the lever is in equilibrium.

11. At three of the angles of a parallelogram taken in order, like parallel forces of 35, 15, and 35 lb. act. Find the force which, acting at the fourth angle parallel to these, will cause the centre of the four forces to be at the point of intersection of the diagonals of the parallelogram.

12. Find the C.G. of two heavy particles, whose weights are 10 lb. and 26 lb., situated at the ends of a thin weightless wire, 4 ft. 6 in. long.

13. Find the C.G. of two spheres, weighing 6 lb. and 5 lb. respectively, the distance between their centres being 4 ft. 7 in.

14. The distance of the C.G. of two heavy particles from the greater is 10 in., the weights of the particles being 4 lb. and 10 lb. Find the distance between them.

15. Two heavy particles, weighing 6 oz. and 10 oz., are attached to the ends of a straight uniform rod 8 in. long and weighing 4 oz. Find the C.G. of the system.

16. Masses of 1 lb., 1 lb., 2 lb., and 2 lb. are placed at the corners A, B, C, D of a square ABCD. Determine, by the principle of "symmetry about a line," a straight line in which the C.G. of the four masses lies, and find its position on that line.

17. Equal masses are placed at the angular points of a regular hexagon. Show that their C.G. is at the centre of the circumscribing circle.

CHAPTER XXV.

DETERMINATION OF THE CENTRE OF GRAVITY.

284. In the last chapter we showed how the position of the C.G. of certain figures—such as the straight line, sphere, and parallelogram—can be inferred from the symmetry of the figures. In most cases, however, it is necessary to divide the body into a number of parts, to deduce the position of its C.G.

DEFINITION.—The medians of a triangle are the straight lines joining its vertices to the middle points of its opposite sides.

In finding the C.G. of a triangle, it will be necessary to assume the following :—

LEMMA. *Any straight line, parallel to the base of a triangle and terminated by its sides, is bisected by the median through the vertex opposite the base.*

This is usually proved by the aid of similar triangles; it can, however, be taken for granted, or proved by Euclid, Book I., thus :—

Let ABC be the triangle, bc be any line parallel to BC. Let D be the middle point of BC, and let AD cut bc in d. Then shall d be the middle point of bc.

For, if not, from dc, cut off $dk = bd$. Join Ak, and join D to b, c, k.

Then, since $BD = DC$; $\therefore\ \triangle ABD = \triangle ACD$ and $\triangle bBD = \triangle cCD$,

$$\therefore \text{ remaining } \triangle AbD = \triangle AcD. \text{ And since } bd = dk,$$

$$\therefore\ \triangle Abd = \triangle Akd, \text{ and } \triangle Dbd = \triangle Dkd; \quad \therefore\ \triangle AbD = \triangle AkD;$$

$$\therefore\ \triangle AcD = \triangle AkD;$$

$\therefore$ ck is parallel to AD, which is impossible, since ck cuts AD in d.

285. To find the C.G. of a triangular area.

Let ABC be a triangular lamina, D, E, F the middle points of the sides. Join AD. Then we shall show that the C.G. of the triangle lies in AD.

Suppose the triangle to be divided into a very large number of thin strips such as *bc*, by drawing straight lines parallel to the base **BC**.

If the strips are made *sufficiently thin*, each may be treated as a uniform thin rod, and the C.G. of such a rod is at its middle point *d*.*

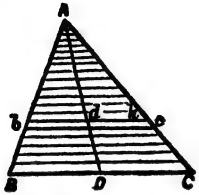


Fig. 140.

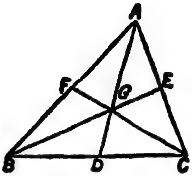


Fig. 141.

But, by the above lemma, the median **AD** passes through *d*. Hence the C.G. of every thin strip of the triangle lies in **AD**. And since the weight of each strip acts as if it were concentrated at its C.G., we see that the C.G. of the whole triangle is the same as that of a certain distribution of weights along the line **AD**.

Therefore the C.G. of the triangle lies in **AD**.

Similarly, by dividing the triangle up into strips parallel to **AC**, it may be shown that the C.G. of the triangle lies in the median line **BE**, and also in **CF**.

Therefore the C.G. *of the triangle is at* **G**, *the common point of intersection of the medians* **AD, BE, CF**.

COR. *The three medians* **AD, BE, CF** *all pass through one common point.* This is a well-known geometrical theorem.

286. The C.G. of a triangular area coincides with the C.G. of three equal particles placed at its vertices.

* We may suppose the triangle to be built up of a number of thin laths or strips of material, each slightly longer than the next above, and fixed side by side. Strictly speaking, their ends would have to be smoothed off along the sides **AB, AC**; but we suppose the strips so thin that the amount smoothed off is inappreciable.

Also it is the point of trisection of any median line which is the more remote from the corresponding vertex.

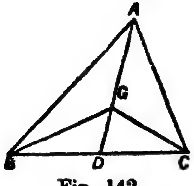


Fig. 142.

(i) Let three equal weights w be placed at **A**, **B**, **C**.

Then, if **D** be the middle point of **BC**, the weights w at **B** and w at **C** have a resultant $2w$ at **D**.

Hence the C.G. of the three weights is also the C.G. of weights $2w$ at **D** and w at **A**.

Therefore it lies in **AD**.

Similarly, it lies in the other two medians.

Therefore it is at **G**, their point of intersection.

That is, it coincides with the C.G. of the triangular area **ABC**.

(ii) Again, since **G** is the point of application of the resultant of weights $2w$ at **D** and w at **A**,

therefore **G** divides **AD**, so that

$$w \cdot \mathbf{AG} = 2w \cdot \mathbf{GD}.$$

$$\therefore \quad \mathbf{AG} = 2\mathbf{GD},$$

and

$$\mathbf{AD} = \mathbf{AG} + \mathbf{GD} = 3\mathbf{GD}.$$

$$\therefore \quad \mathbf{GD} = \tfrac{1}{3}\mathbf{AD}, \text{ and } \mathbf{AG} = \tfrac{2}{3}\mathbf{AD}.$$

Therefore **G** is that point of trisection of **AD** which is the more remote from the vertex **A**.

COR. 1. If each of the weights w is *one-third* the weight of the lamina, their total weight and the position of their C.G. will be the same as for the lamina. Hence *a uniform triangular lamina is statically equivalent to three equal weights placed at its vertices, each being one-third the weight of the lamina.*

This theorem is often useful.

COR. 2. *The point of intersection of the three medians of a triangle is one of the points of trisection of each of them.*

This may also be proved by pure geometry. The proof is left as an exercise for the reader.

287. Having given the weights and centres of gravity of different parts of the body, to find the C.G. of the whole body.

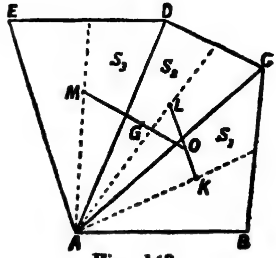


Fig. 143.

Let S_1, S_2, S_3 be different parts of a body, and let w_1, w_2, w_3 be the weights, K, L, M the C.G.'s, of S_1, S_2, S_3 respectively. It is required to find the C.G. of the whole body made up of the parts S_1, S_2, S_3.

The weights w_1, w_2 may be taken to act at K, L. Therefore their resultant is a weight $w_1 + w_2$ acting at a point O on KL, such that

$$w_1 . \mathrm{KO} = w_2 . \mathrm{OL},$$

whence O can be found.

Thus O is the C.G. of the body made up of S_1 and S_2.

The weight of S_3 is w_3 acting at M. Therefore the weight of the whole body is $w_1 + w_2 + w_3$ acting at a point G, such that

$$(w_1 + w_2)\ \mathrm{OG} = w_3 . \mathrm{GM},$$

whence G can be found.

Thus G is the C.G. of the body made up of S_1, S_2, S_3.

In like manner, if the weights and C.G.'s of four or more parts of a body are known, we can find the C.G. of the whole body. The method is identical with that used for finding the centre of parallel forces (§ 271).

Examples.—(1) To find the C.G. of a cubical box without a lid.

Let a be the length of a side of the cube. Let O be the centre of the cube, A the centre of its base, G the required C.G. (figure should be drawn). Then it is easy to see that the C.G. of the four sides of the box is at O, and their total area is $4a^2$. Also the C.G. of the base is at A, and its area is a^2. Therefore G is the C.G. of weights a^2 at A and $4a^2$ at O.

$$\therefore\ a^2 \times \mathrm{AG} = 4a^2 \times \mathrm{GO}.$$
$$\therefore\ \mathrm{AG} = 4\mathrm{GO}, \text{ and } \mathrm{AO} = 5\mathrm{GO}.$$
$$\therefore\ \mathrm{GO} = \tfrac{1}{5}\mathrm{AO}, \text{ and } \mathrm{AG} = \tfrac{4}{5}\mathrm{AO}.$$

But $\mathrm{AO} = \frac{1}{2}a$. Therefore $\mathrm{AG} = \frac{2}{5}a$.

Hence the C.G. is $\frac{2}{5}$ the height of the cube above the base.

(2) To find the C.G. of a wire bent into an angle.

Let a uniform wire **AB** be bent into an angle **ACB**. We may consider it as consisting of two straight wires **AC**, **CB** joined together at **C**.

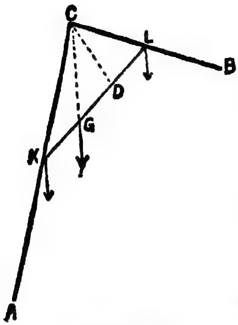


Fig. 144.

The C.G.'s of these two portions are at their middle points **K**, **L**, and their weights are proportional to their lengths **AC**, **CB**. Hence the C.G. of the whole wire is at a point **G** in **KL**, such that

$$AC \times KG = CB \times GL,$$

or

$$\frac{GK}{GL} = \frac{CB}{CA}.$$

288. To find the C.G. of the area of any polygon.

Divide the polygon into triangles (S_1, S_2, S_3, Fig. 143) by joining one of its vertices **A** to the other vertices not already joined to it. Find **K**, **L**, **M**, the C.G.'s of these triangles. Then the required C.G. of the polygon is the C.G. of weights at **K**, **L**, **M** proportional to the areas S_1, S_2, S_3, and is therefore given by the construction of § 287.

CAUTION.—The C.G. of a polygon is not, in general, the C.G. of *equal weights* placed at all the corners of the polygon.

289. To find the C.G. of a portion of a body.

Having given the weight and C.G. *of a whole body and of any part removed from it, to find the position of the* C.G. *of the remaining part.*

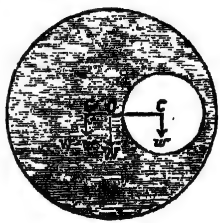


Fig. 145.

Let **O** be the C.G. of a body, W its weight.

Let **C** be the C.G. of any part of the body, w its weight.

Let **G** be the C.G. of the remainder of the body. It is required to find **G**.

The weight of this remaining part is evidently $W - w$.

Now, since **O** is the C.G. of the whole body, **O** is the C.G. of weights w at **C** and $W - w$ at **G**.

$$\therefore \quad \mathbf{C}, \mathbf{O}, \mathbf{G} \text{ lie in a straight line, and}$$

$$(W - w)\,\mathbf{GO} = w \,.\, \mathbf{OC}.$$

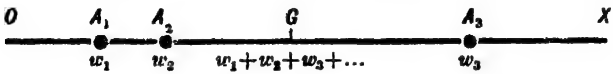
Therefore **G** lies on **CO** produced through **O**, so that

$$\mathbf{OG} = \frac{w}{W - w}\,\mathbf{CO}.$$

[For illustrative examples see below.]

290. To find the C.G. of a number of weights at points in a straight line.

When the C.G.'s of different parts of a body, or series of weights, all lie in a straight line, the C.G. of the whole may be found by the formula of § 274. If $w_1, w_2, w_3, \ldots$ be the weights, $x_1, x_2, x_3, \ldots$ the distances of their points of



application from a point **O** in the line, the distance of their C.G. from **O** is $\bar{x}$, where

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Examples.—(1) **ABCD** is a square of paper, and **E**, **F** are the middle points of **AB**, **AD**.

(i) To find the C.G. of the portion left when the triangle **AEF** is cut off.

(ii) To find the C.G. of the whole when the triangle **AEF** is doubled over.

Let **O** be the centre of the square, and let $2a$ be the length of a diagonal so that $\mathbf{OA} = a$. Let W be the weight of the square; then that of $\triangle\mathbf{AEF}$ is $\frac{1}{8}W$.

(i) Now the C.G. of $\triangle\mathbf{AEF}$ lies at **K** where $\mathbf{AK} = \frac{1}{3}\mathbf{AO}$, $\mathbf{OK} = \frac{2}{3}a$. Hence, if **G** be the C.G. of the remaining portion,

$$\tfrac{7}{8}W \times \mathbf{OG} = \tfrac{1}{8}W \times \mathbf{OK}.$$

$$\therefore \quad \mathbf{OG} = \tfrac{1}{7}\mathbf{OK} = \tfrac{2}{21}a = \tfrac{1}{21}\mathbf{AC}.$$

(ii) When $\triangle$ **AEF** is folded to the position **OEF** (Fig. 146) its c.g. is at **L** where $\mathbf{OL} = \frac{1}{3}a$. Let **H** be the c.g. of the whole. Taking moments about **O**, we have to find the centre of $\frac{7}{8}W$ at **G** and $\frac{1}{8}W$ at **L**; hence, by § 290 or § 274,

$$W \times \mathbf{OH} = \tfrac{7}{8}W \times \mathbf{OG} - \tfrac{1}{8}W \times \mathbf{OL};$$

$$\therefore \quad \mathbf{OH} = \tfrac{7}{8}\mathbf{OG} - \tfrac{1}{8}\mathbf{OL} = \tfrac{1}{12}a - \tfrac{1}{24}a = \tfrac{1}{24}a = \tfrac{1}{48}\mathbf{AC}.$$

(2) To find the c.g. of a hollow spherical bullet 3 cm. in diameter containing an excentric spherical cavity 1 cm. in diameter, whose centre is 8 mm. distant from the centre of the bullet.

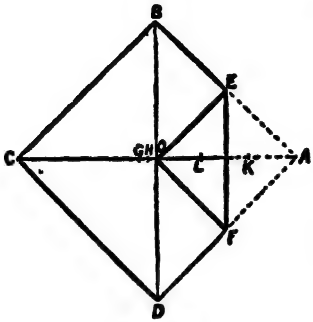


Fig. 146.

Let **O** be the centre of the surface of the bullet, **C** the centre of the cavity (Fig. 145).

The volumes of the bullet and of the cavity are respectively proportional to the cubes of their diameters, that is, as 27 to 1.

Hence, if W denote the weight which the bullet would have if no cavity existed, w the weight of matter which would fill the cavity, then $W = 27w$, and the weight of the actual bullet is

$$W - w = 26w.$$

Hence the required c.g. of the hollow bullet is a point **G** in **CO** produced, such that

$$\mathbf{OG} = \mathbf{CO} \times \frac{w}{W - w} = 8 \text{ mm.} \times \tfrac{1}{26} = \tfrac{4}{13} \text{ mm.}$$

(3) In a three-draw telescope, the lengths of the respective cylinders are 6, 7, and 8 inches, and their weights are proportional to the squares of their lengths. Find the position of the c.g. when fully drawn out.

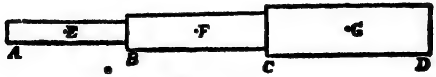


Fig. 147.

Let **AB**, **BC**, **CD** be the three cylinders, **E**, **F**, **G** their separate centres of gravity [§ 283 (5)]. By symmetry, the c.g. of the whole will lie somewhere on the central axis of the telescope. Let x denote its distance from **A**. The weights being denoted by 36, 49, and 64, we have

$$x = \frac{36.3 + 49.(6 + 3\frac{1}{2}) + 64.(6 + 7 + 4)}{36 + 49 + 64} = \frac{108 + 465\frac{1}{2} + 1088}{149}$$

$= 11{\cdot}151$ inches.

Thus the c.g. is about 2 inches from **C** towards **A**.

[The following example (*which the student should illustrate with a figure*) affords an instructive introduction to the method of the next article.]

(4) To find the c.g. of weights of 2, 3, 4, 5 lb., placed at the four corners of a square slab **ABCD**.

Let x be the distance of the c.g. from **AD**, y its distance from **AB**, a the length of the side of the square.

The total weight $= 2 + 3 + 4 + 5$ lb. $= 14$ lb.

Place the square with **AD** vertical and **AB** horizontal. Then the lines of action of the weights at **C**, **D** will pass through **B**, **A**, respectively, and the vertical through **G** (the c.g.) will cut **AB** at a distance x from **A**. Therefore the equation of moments about **A** gives

$$14x = 2.0 + 5.0 + 3.a + 4.a = 7a;$$

$$\therefore \quad x = \tfrac{1}{2}a.$$

Making **AB** vertical, we have, in like manner,

$$14y = 2.0 + 3.0 + 4.a + 5.a = 9a;$$

$$\therefore \quad y = \tfrac{9}{14}a.$$

Hence **G** lies on the bisector of the sides **AB**, **DC** at a distance $\frac{9}{14}a$ from **AB**.

***291. To find the C.G. of any number of weights at given points in one plane.**

Let any number of particles of known weights $w_1, w_2, w_3, \ldots$ be situated at given points $A_1, A_2, A_3, \ldots$ in one plane. Draw two straight lines **OX**, **OY** at right angles to one another in the plane, and let the distances of each weight from each of these two lines be measured.*

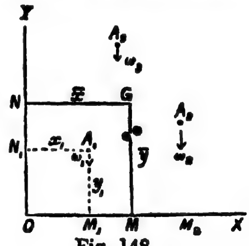


Fig. 148.

Let $x_1, x_2, x_3, \ldots$ be the distances of the weights from **OY**; $y_1, y_2, y_3, \ldots$ their distances from **OX**.

[So that if, from any weight A_1, perpendiculars A_1M_1 on **OX** and A_1N_1 on **OY** be drawn, we have

$$x_1 = OM_1 = N_1A_1 \text{ and } y_1 = ON_1 = M_1A_1.]$$

Let **G** be the required c.g. of the weights. Draw **GM**, **GN** perpendicular on **OX**, **OY**, and let $\bar{x} = OM = NG$, $\bar{y} = ON = MG$.

The resultant of the weights $w_1, w_2, w_3, \ldots$, acting at $A_1, A_2, A_3, \ldots$, is $w_1 + w_2 + w_3 + \ldots$ acting at **G**.

* We may suppose the "particles in one plane" to be a number of small weights attached to a flat square sheet of cardboard, and the two straight lines at right angles to be two adjacent edges of the square.

Suppose the plane turned so that OY is vertical and OX horizontal. Then, since the sum of the moments of the several weights about O is equal to the moment of their resultant, and the arm of w_1 is OM_1,

$$\therefore \quad OM \times (w_1 + w_2 + w_3 + \ldots) = OM_1 \times w_1 + OM_2 \times w_2 + OM_3 \times w_3 + \ldots,$$

or $$\bar{x} = OM = \frac{w_1 x_1 + w_2 x_2 + w_3 x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Turn the figure round till OX is vertical and OY horizontal.

By taking moments in like manner about O, we have

$$ON \times (w_1 + w_2 + w_3 + \ldots) = ON_1 \times w_1 + ON_2 \times w_2 + ON_3 \times w_3 + \ldots,$$

whence $$\bar{y} = ON = \frac{w_1 y_1 + w_2 y_2 + w_3 y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Hence the distances OM, ON are known, and by completing the rectangle OMGN, the position of G, the required C.G., can be found.

Examples.—(1) To find the distances of the C.G. of a right-angled triangle from the two sides containing the right angle.

Replacing the triangle by three equal weights at its vertices (§ 286, Cor. 1), the formula readily gives

$$\bar{x} = \tfrac{1}{3}a, \quad \bar{y} = \tfrac{1}{3}b,$$

where a, b are the sides parallel to the perpendiculars $\bar{x}$, $\bar{y}$.

(2) To a square ABCD, the half-square DCE is applied. Find, algebraically, the position of the C.G. of the whole quadrilateral thus formed.

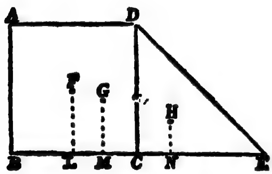


Fig. 149.

Let a side of the square $= a$, so that CE also $= a$.

Let F, H, G be the C.G.'s of the square, the triangle, and the quadrilateral.

Draw FL, GM, HN perpendiculars on BE. Then it is clear that

$$BL = LF = \frac{a}{2} \text{ and } CN = NH = \frac{a}{3}$$

Thus $$BN = \frac{4a}{3}.$$

The areas of the square, triangle, and quadrilateral are

$$a^2, \ \frac{a^2}{2}, \ \frac{3a^2}{2}, \text{ respectively.}$$

Hence, by § 291,

$$\bar{x} = BM = \left(a^2 \times \frac{a}{2} + \frac{a^2}{2} \times \frac{4a}{3}\right) \div \frac{3a^2}{2} = \left(\frac{a}{2} + \frac{2a}{3}\right) \div \frac{3}{2} = \frac{7}{9}a,$$

and $$\bar{y} = MG = \left(a^2 \times \frac{a}{2} + \frac{a^2}{2} \times \frac{a}{3}\right) \div \frac{3a^2}{2} = \frac{4}{9}a.$$

The position of G is thus accurately determined.

292. Work Done in Raising Weights.—*The work done in lifting a number of weights off the ground or raising them up to the ground is the same as if their total weight were collected at their* C.G.

Let there be any number of weights $w_1, w_2, w_3, \ldots$, and let them be raised from the ground to heights $x_1, x_2, x_3, \ldots$. Let W be their total weight, $\bar{x}$ the height of their C.G. in the new position.

Then the work done in raising the weights

$$= w_1x_1 + w_2x_2 + w_3x_3 \ldots.$$

But, by the last article,

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{W};$$

$$\therefore \quad w_1x_1 + w_2x_2 + w_3x_3 + \ldots = W \cdot \bar{x};$$

$\therefore$ whole work done $= W\bar{x} =$ work required to lift the total weight W to the height of the C.G.

Similarly, by taking $x_1, x_2 \ldots$ to represent the depths of a number of weights *below* the ground, we see that the work done in lifting a number of weights from below the surface to the ground is the same as if the weights were all concentrated at their C.G.

Examples.—(1) The lifting work done in building a cylindrical tower is the same as would be required to lift the whole of the materials through $\frac{1}{2}$ the height of the tower.

(2) The work done in digging a ditch of triangular section through earth of uniform material is the same as would be required to lift the total mass of earth through $\frac{1}{3}$ the depth of the lowest point of the ditch (so far as lifting the material is concerned).

293. To find the C.G. of four equal weights placed at the corners of a triangular pyramid.

Let **ABCD** be the pyramid, **H** the C.G. of its base **ABC**.

Then the C.G. of four equal weights w at **A, B, C, D** is the same as that of w at **D**, and $3w$ at **H** (since **H** is the C.G. of the weights at **A, B, C**). Therefore **G**, the C.G. of the weights, is a point in **DH**, such that

$$3\mathbf{GH} = \mathbf{DG}, \text{ or } \mathbf{DG} : \mathbf{GH} = 3 : 1,$$

whence $$\mathbf{GH} = \tfrac{1}{4}\mathbf{DH}, \quad \mathbf{DG} = \tfrac{3}{4}\mathbf{DH}.$$

294. The C.G. of a Pyramid or Cone.—We may here state that the C.G. of a triangular pyramid coincides with the C.G. of four equal particles placed at its four vertices. It therefore

(§ 293) divides the line joining the vertex to the C.G. of the base into parts which are as 3 : 1.

In like manner, if the base of a pyramid has *any number of sides*, its C.G. is in the line joining the vertex to the C.G. of the base, and at a distance from the latter point equal to $\frac{1}{4}$ the distance of the vertex.

Again, what holds good for the C.G. of a pyramid will also hold good for the C.G. of a cone.

In a *right circular cone* (the only kind of cone we have to consider) the C.G. is, therefore, in the *axis*, at a distance from the base of $\frac{1}{4}$ the altitude of the cone.

But the C.G. of the curved surface of a right circular cone is on its axis, at a distance from the base equal to $\frac{1}{3}$ of the altitude.

EXAMPLES XXV.

1. Find the C.G. of a bent wire ABC, the arm BC being twice the length of the arm AB.

2. A uniform isosceles triangle has its two equal sides each 5 ft. long and its base 8 ft. long; find its C.G. If its weight be 5 lb., and a weight of 10 lb. be hung at the vertex, find the C.G. of the whole.

3. The mass of an equilateral triangle is 4 lb. Masses of 1, 1, and 2 lb. respectively, are placed at its angular points. Find the C.M. of the system.

4. Find the load which must be placed at the corner of an equilateral triangular plate to bring the C.G. to the middle of the median through that corner.

5. If at one angle A of a triangular lamina ABC be placed a weight equal to the weight of the triangle, where will be the C.G.?

6. Weights of 2, 3, 5, and 6 lb. are placed at the corners A, B, C, D of a weightless square lamina ABCD, whose side is 12 in. Find the position of the C.G. of the weights.

7. Weights of 2, 4, 5, and 3 lb. are suspended from the corners A, B, C, D of a horizontal square board ABCD without weight, whose side is 2 ft. Find the position of the C.G. of the weights. Would it change if the board were inclined to the horizontal?

8. Find the C.M. of five equal heavy particles placed at the angular points A, B, C, D, E of a regular hexagon ABCDEF.

9. Find the C.G. of seven equal particles placed at the corners of a regular octagon.

*10. Weights of 4, 2, 5, and 3 lb. are placed at the corners A, B, C, D of a parallelogram ABCD. If a weight of 14 lb. be placed at O, the intersection of the diagonals, find the C.G. of the system.

11. A square ABCD is divided into four triangles by its diagonals, which intersect in O. Find the C.G. of the area left when one of the triangles AOB is removed.

12. If at B and C, two of the angles of a triangular lamina ABC, weights equal to half the weight of the lamina be placed, where will be the common C.G. of the weights and the lamina?

13. If G, the C.G. of a triangle ABC, be joined to the extremities of the side BC, and the triangle BGC be removed, find the C.G. of the rest of the triangle.

14. Isosceles triangles ABC, DBC, the former of which is double of the latter, are described on the same side of the same base BC. Find the C.G. of the area included between their sides.

15. Two uniform cylinders of the same material are joined together, end to end, so that their axes are in the same straight line. One of the cylinders is 12 in. long and 3 in. in diameter, the other is 18 in. long and 2 in. in diameter. Find the C.G. of the combination.

16. One corner of a square ABCD is cut off by a straight line passing through the middle points of two adjacent sides AB and BC. Find the C.G. of the remaining portion of the square.

*17. Five pieces of a uniform chain are hung at equidistant points of a rigid rod without weight, and their lower ends are in a straight line passing through one end O of the rod. Prove that the C.G. is in the line joining O to C, the middle point of the longest chain, and at a distance from O equal to $\frac{11}{15}$OC.

18. A telescope consists of three tubes, each 10 in. in length, sliding one within another, and their weights are 8, 7, and 6 oz. Find the position of the C.G. when the tubes are drawn out to their full extent.

19. A uniform square sheet of paper ABCD has its two adjacent corners A and B folded over so as to coincide with its centre O. Find its C.G.

20. If the two corners folded over in the last question be torn off, where is now the C.G. of the remainder of the square?

21. ABCD is a square, O the point of intersection of its diagonals, E, F the middle points of the adjacent sides AB, AD. Find the C.G. when the square AEOF is removed.

22. The longer side BC of a rectangle BCDE is $\sqrt{3}$ times the shorter, and on the longer side an isosceles triangle ABC, having the angle BAC equal to a right angle, is described. Find the C.G. of the lamina made up of the rectangle and the triangle.

*23. A cylinder of metal is 1 ft. high, 6 in. external radius, and 5 in. internal radius, and 11 in. deep inside; it is open at the top. Find the position of its C.G.

24. A uniform rod AB is 4 ft. long and weighs 3 lb. Weights of 1, 2, 3, 4, and 5 lb., respectively, are attached at intervals of 1 ft., the smallest weight being at the end A. Find the distance of the C.G. of the system from A.

25. Five masses of 1, 2, 3, 4, 5 oz. respectively, are placed upon a square table ABCD; their distances from the edge AB are 2, 4, 6, 8, 10 in., and from the adjacent edge BC, 3, 5, 7, 9, 11 in. respectively. Find the distances of their centre of mass (C.G.) from the two edges.

26. Weights of 1, 2, 3, and 4 lb. are placed at the angular points A, B, C, D respectively, of a square ABCD. Find the distance of the C.G. of the system from the centre of the square.

27. Find the number of foot-pounds of work required to wind up a chain 50 ft. long and weighing 120 lb., which hangs by one end.

28. A shaft, 280 ft. deep and 15 ft. in diameter, is full of water; how many foot-pounds of work are required to empty it? [Weight of 1 cub. ft. of water = 1000 oz.]

29. A well is to be made 40 ft. deep and 3 ft. in diameter. Find the work done in raising the material, supposing a cubic foot of it weighs 140 lb.

30. Find the C.G. of a circular board from which a circular piece has been cut out, having as diameter a radius of the board.

31. Find the C.G. of half a hexagon, bounded by three sides and a diagonal.

32. A piece of uniform paper in the form of a regular hexagon has one of the equilateral triangles obtained by joining the centre to two consecutive angular points cut out. Determine the position of the C.G. of the remainder.

33. A uniform plate of metal 10 in. square has a hole 3 in. square cut out of it, the centre of the hole being $2\frac{1}{2}$ in. distant from the centre of the plate. Find the position of the C.G. of the plate.

34. A weight equal to that of a triangular pyramid is placed at one corner of the pyramid. Find the C.G. of the system.

EXAMINATION PAPER XIII.

1. Define the *centre* of a system of parallel forces, and state a method of finding its position when there are more than two forces.

2. AB and CD are two uniform beams each 12 ft. long, of the same weight 120 lb., and resting in horizontal positions. The ends A, B, D rest on supports, and the end C rests on the rod AB at a point 4 ft. from A. Find the pressure on each support.

3. A number of coins lie upon a square table. If the masses of the coins are known, as well as their distances from two adjacent sides of the table, show how to determine the position of their common C.M.

4. Two men carry a weight of 200 lb. on a pole between them. If they are 5 ft. apart and the weight is slung at a distance of 2 ft. from one of the men, what part of the weight will the other man bear?

5. Find the centre of gravity of a uniform triangular pyramid, having given that it is the same as that of four equal heavy particles placed at the four corners of the pyramid.

6. Find the centre of five like parallel forces of 1 lb., 2 lb., 3 lb., 2 lb., and 1 lb., acting at points A, B, C, D, E in a straight line, such that AB, BC, CD, DE equal 1, 2, 3, and 4 in. respectively.

7. Having given the C.G. of a body and that of one part, find the C.G. of the remainder.

8. Show that the C.G. of a uniform triangular area coincides with that of three equal heavy particles placed at its angular points.

9. Two circles whose radii are 6 in. and 4 in. are drawn on a uniform sheet of paper and touch one another internally. The portion bounded by the circumferences of the two circles is cut out; find its C.G.

10. Find the C.G. of four heavy particles weighing 2, 3, 4, and 5 lb. respectively, placed at the corners A, B, C, D of a square whose side is 14 in. long.

CHAPTER XXVI.

PROPERTIES OF THE CENTRE OF GRAVITY.

295. Equilibrium of a Heavy Body about a Fixed Point.

If a heavy body be at rest when supported at one point, and be not acted upon by forces other than gravity, the C.G. *and the point of support will be in the same vertical line.*

For the only forces acting on the body are—

(i) The resultant force due to gravity. This is the weight of the body, and acts vertically through the C.G.

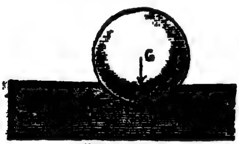


Fig. 150.

(ii) The reaction at the point of support.

These forces are in equilibrium; therefore they must be equal and opposite and in the same straight line.

That is, the point of support must be in the vertical through the C.G.

This refers to a body either resting on a surface at one point, as in Fig. 150, or capable of rotation about a point, or suspended by a string as in Figs. 151, 152.

296. The **plumb-line,*** used by builders and others for finding the direction of the vertical, is a flexible string from which a heavy weight of lead, or **plummet**, hangs. When at rest, the string is vertical throughout its length, for the tension at any point has only to support the weight of the plummet, which acts *vertically* (§ 275). We may sometimes define the *vertical* line *at any point* as the direction of a plumb-line at that point.

* From the Latin *plumbum* = "lead."

297. To find by experiment the C.G. of a lamina.

The C.G. of a sheet of metal or any other material, or a wooden plank, or a wire of any shape, may be found in the following manner :—

Suspend the body by a string attached to any point **A**, and on it draw the vertical line **AD** through **A**. This line may be traced either by means of a plumb-line or by producing the direction of the string. We know, by § 295, that the C.G. lies in **AD**.

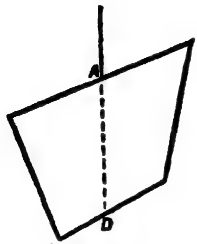


Fig. 151.

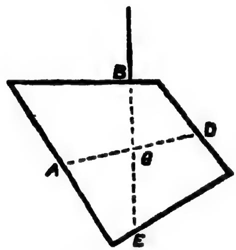


Fig. 152.

Now suspend the body from any other point **B**, and on it draw the vertical line **BE** through **B**. As before, the C.G. lies in **BE**. Therefore the required C.G. is at the intersection of **AD, BE.**

If we now hang the lamina from a third point **C**, it will be found that **G** is vertically below **C**, thus verifying that **G** is the C.G.

This method is *theoretically* applicable to any body whatever, but except in the case of a lamina, or a wire bent in one plane, it would usually not be very easy to mark the two positions of the vertical, **AD** and **BE.**

If the lamina is in the shape of a rectangle or parallelogram, and is hung up from one of its angles, the diagonal through that angle will be found to be vertical, thus verifying that the C.G. lies in the diagonals. *See also* § 330.

298. To apply the method to a straight thin rod (*e.g.* a ruler, a straight walking-stick, or, better still, a billiard cue), we suspend it by means of two very fine strings from any

point **O**. Also make a plumb-line by hanging a small weight **W** by a single thread from **O**. If the thread **OW** cuts the rod in **G**, the C.G. of the rod will be at **G**. Mark this point on the rod. Then, if the rod be laid on the finger or on any support touching it at **G**, it will balance.

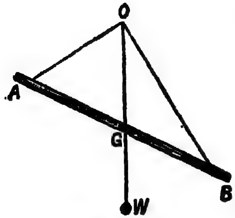


Fig. 153.

If the rod is uniform, **G** will be found to be at its middle point. If one end of the rod is heavier than the other, **G** will be found to be nearer to the heavier end.

Example.—A heavy rod **AB**, whose centre of gravity **G** is not at its middle point, is suspended by a single string **AOB** over a *smooth* peg **O**. Discuss the possible positions of equilibrium.

[The student should draw a figure very like Fig. 153, but making **AG** rather shorter, so that **OG** bisects the angle **AOB**.]

Since the weight of the rod at **G** acts vertically downwards, **OG** must be *vertical*.

Moreover, since the peg is smooth, the tension T is the same in each part of the string.

Hence the resultant of the two tensions T must bisect **AOB**, the angle between them. For equilibrium, this resultant is equal and opposite to the weight, and must act along **OG**. Thus **OG** must bisect the angle **AOB**, that is, *the strings must be equally inclined to the vertical.*

By Euclid VI. 3, $\therefore \dfrac{AO}{OB} = \dfrac{AG}{GB}$. [Appendix, § 10.]

That is, *the parts of the string must be proportional to the segments of the rod.*

By drawing **GE** parallel to **BO**, we get a convenient triangle of forces, **OEG**, from which the relation between the tension and the weight can be calculated, in particular cases.

There are other less interesting positions of equilibrium, namely, with the rod vertical, and either end uppermost. If we assume the peg to be of insignificant dimensions, the three forces will then act in one straight line, and the tension will be equal to half the weight.

It is noteworthy that, in the case of a picture-frame as ordinarily suspended, the two parts of the string would be equally inclined to the vertical, even if the frame itself were hanging crookedly, if the peg were perfectly smooth.

299. If a heavy body is supported *at* its C.G., it will balance in every position.

Since the C.G. is at the point of support, the body's weight always acts through the point of support. This weight cannot move the point of support, because that point is fixed; and it cannot turn the body round, because it has no moment about that point. Hence the body must remain balanced in every position.

In consequence of this property, the C.G. *is very commonly spoken of as the* **balancing point** *of a body.*

300. DEFINITION.—When a body rests upon any hard flat surface, its **base** is defined to be the area enclosed by a fine string drawn tightly round it so as to enclose all points of the body in contact with the supporting surface.

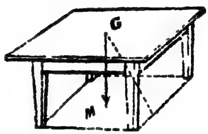


Fig. 154.

When a glass tumbler rests on a table, the *base* of the tumbler is evidently the circular area enclosed by the parts of the tumbler touching the table. But in the case of a table resting on the floor on its four legs, the *base* is the quadrilateral formed by joining the feet, for this would be the figure assumed by a string pulled tightly round the points of contact. The definition ensures that when the body is overturned it must turn about some point or line bounding the base.

301. When a body is placed on a plane it will stand or fall according as the vertical line through its C.G. falls within or without the base on which it rests.

(i) First let the vertical through the C.G. cut the supporting plane at a point **M** *outside* the base **AD**; let **A** be the point of the base that is nearest to **M**. Then the weight of the body is equivalent to a single resultant force acting at **G**, and the moment of this weight (Fig. 155) tends to turn the body about **A** in such a way as to lift up all the other points of the body touching the plane. Now the reaction of the plane always acts away from it, and never tends to prevent any part of the body from being lifted off. Hence there is nothing to counteract the tendency of the body to overturn about **A**. Therefore it will *fall.*

(ii) Second, let the vertical through the C.G. cut the supporting plane at a point **M** inside the base **AD**. Then, if **A** be any point at the edge of the base, the moment of the weight about **A** tends to press down the other points of the body touching the plane (Fig. 156), as at **D**. Now the reaction of the plane prevents the body from penetrating it. Hence, the body will *stand*. And if it be slightly tilted up, its weight will bring it back to its original position as soon as it is let go.

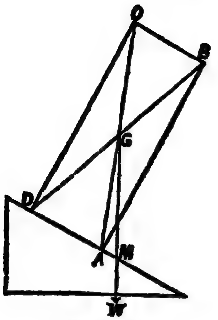


Fig. 155.

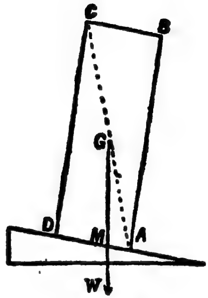


Fig. 156.

As a particular case, when a body rests touching the ground at a single point, the vertical through the centre of gravity must pass through that point (Fig. 150).

Observation.—The plane supporting the body may be either horizontal or inclined, provided that it is sufficiently rough to prevent the body from sliding down.

Illustrations.—(1) A cart or tricycle will overturn if the vertical through its C.G. falls outside the wheel base (Fig. 157).

(2) A porter carrying a heavy trunk in one hand often extends the opposite arm at full length in order to more readily bring his C.G. over a point between his two feet. A man carrying a heavy weight in front of him leans back in order to bring his C.G. over his base.

302. **Experimental Verifications.**—The reader is strongly recommended to test the truth of the theorem by some simple experiments made with any common objects around him.

Thus, for example, if any body of rectangular section **ABCD** (say a brick) be stood upon a rough plank, and the plank be gradually tilted about the edge **A**, the body will remain standing as long as its C.G. does not overlap the base, but directly this happens it will fall over about its lowest edge Figs. 155, 156)

Fig. 157.

Now the C.G. of a rectangular body lies in the diagonal plane **AC**. Hence the body will overturn just after the diagonal **AC** has passed through the vertical position.

Again, a book can be placed on the top of another book resting on a table, provided that the middle (or C.G.) of the upper book does not project beyond the lower book (*i.e.* beyond the *base*). Hence, in order to make the upper book project beyond its middle, weights must be placed on the supported end to bring the C.G. nearer that end.

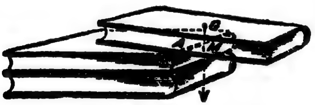


Fig. 158.

Fig. 159.

303. Stable, Unstable, and Neutral Equilibrium.—Theoretically, it is possible to balance a body by supporting it at a point either vertically *above* or vertically *below*, or *at* its C.G. But, practically, it is often very difficult to keep a body balanced on a point even for a short time, and the least disturbance suffices to overturn it. Bodies have, as we know, a tendency to fall into certain natural positions of equilibrium, and to fall away from other positions.

DEFINITION.—Bodies are said to be in **unstable** equilibrium when, after a slight disturbance, they tend to move further and further away from their equilibrium position, *i.e.* to upset.

Thus an egg naturally rests on a table with its side touching the table But it is difficult to balance the egg on its end; and, if this has been done by bringing the C.G. directly over the point of support, the egg will overturn, as in Fig. 163, with the slightest shake or breath of air, or other disturbance, which moves the egg and therefore its C.G. or its point of support a little to one side or the other. In ordinary language, we express this fact by saying that the egg is *top heavy*.

A pin would theoretically satisfy the condition of equilibrium of § 295 if stood upright with its point resting on a plate. But no hand is sufficiently steady and patient to place it exactly in the right position, nor could the plate and pin remain sufficiently undisturbed for the pin to continue balanced for more than an instant, even if it were so placed. The pin would in fact be *top heavy*.

A walking-stick standing upright on the finger, a ball or marble placed at the top of an inverted bowl, are also examples of unstable equilibrium.

NOTE.—*When we say that a body is "top heavy," we imply that it is in unstable equilibrium.*

DEFINITION.—Bodies are said to be in **stable** equilibrium when they tend to return to their equilibrium position after being slightly disturbed.

Thus, for example, a weight (such as a plummet) hanging from a string will of its own accord fall into a position of equilibrium with the string vertical. If pulled aside, it will at first swing to and fro, but the string will at last again assume a vertical position.

A stick hanging by its crook, a ball or marble inside a basin, the beam of a balance, are also examples of stable equilibrium.

DEFINITION.—Bodies are said to be in **neutral** equilibrium when, after being slightly displaced, they remain in their new position.

A ball whose C.G. is at its centre is in neutral equilibrium when placed on a horizontal surface. For, however the ball be rolled about, its C.G. will always be vertically above its point of contact, and hence it will always remain in equilibrium (Fig. 164, p. 291).

An egg or cylinder resting on its side when allowed to roll along the table, a heavy body of any kind supported *at* its C.G. (§ 299), a door turning on its hinge, are also examples of neutral equilibrium.

A *cone* affords a good illustration of all three kinds of equilibrium. On its base it is *stable*, on its vertex *unstable*, and on its side *neutral* for lateral displacements (rolling).

304. Stability of a Body with One Point Fixed.

A heavy body, moveable freely about a fixed point O, is in stable, unstable, or neutral equilibrium, according as O is vertically above, vertically below, or at G, the body's C.G.

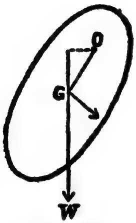


Fig. 160.

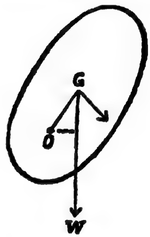


Fig. 161.

For if the line OG is not quite vertical, Figs. 160, 161 show that the moment of the weight acting at G tends to turn the body about O *towards* a position in which G is vertically below

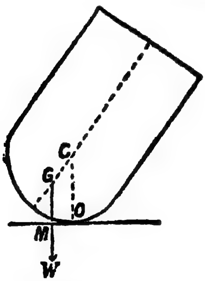


Fig. 162.

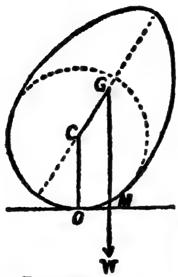


Fig. 163.

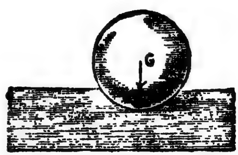


Fig. 164.

O, and *away* from a position in which G is vertically above O. Hence the former position is stable and the latter unstable. And since the body balances in every position when supported *at* G, its equilibrium is then neutral.

305. Stability of a Body resting on a Horizontal Plane.

If the surface of a body is spherical and its C.G. is not at its centre, the body will be in stable equilibrium on a horizontal plane if its C.G. be vertically *below* its centre, and in unstable equilibrium if its C.G. be vertically *above* its centre; for Figs. 162, 163 show that the moment of the weight tends to turn the body *towards* its equilibrium position in the former case, and *away* from it in the latter. If the C.G. is *at* the centre, equilibrium is neutral (Fig. 164).

The whole body need not necessarily be spherical, provided that the part of the surface touching the plane is spherical.

Thus, if a hemisphere of lead is joined to a cylinder of cork so that the C.G. of the whole is below the centre of the hemisphere, it will stand upright although it *looks* very top-heavy. For lead is so much heavier than cork, that a cork of considerable length may be attached to the hemisphere without raising the C.G. above the centre (Fig. 162).

Several toys act on this principle.

EXAMPLES XXVI.

1. A uniform straight lever, 12 ft. long and weighing 12 lb., balances about a certain point when weights of 3 lb. and 12 lb. are suspended from its ends. How far will the point of support be moved when each of the weights at the ends is doubled?

2. A uniform rod, 12 ft. long and 30 lb. in weight, is supported by a prop at one end. Find the force which must act vertically upwards at a distance of 8 ft. from the prop to keep the rod horizontal.

3. A uniform straight bar, 10 in. long and 8 oz. in weight, balances about a certain point when weights of 4 and 8 oz. are suspended from its ends. Equal weights are then added to each of the weights suspended from the ends, and it is found that the fulcrum must be moved $\frac{2}{3}$ in. Find the weight added.

4. Weights of 1, 2, 3, and 4 lb. are suspended from a uniform lever 5 ft. long, at distances of 1, 2, 3, and 4 ft. respectively, from one end. If the mass of the lever be 4 lb., find the position of the point about which it will balance.

5. A rod is hung from a smooth peg by means of a string passing over the peg attached to the extremities of the rod. Find the position of equilibrium when the C.G. of the rod is (i) at its middle point, (ii) not at its middle point.

6. How would you find experimentally the C.G. of a straight, but not uniform, thin rod?

7. A rod 12 ft. long has a weight of 1 lb. suspended from one end, and when 15 lb. are suspended from the other end it balances at a point 3 ft. from that end, while, if 8 lb. are suspended there, it balances at a point 4 ft. from that end. Find the weight of the rod and the position of its C.G.

8. A bar projects 6 in. beyond the edge of a table, and when 2 oz. are hung on to the projecting end the bar just topples over; when it is pushed out so as to project 8 in. beyond the edge, 1 oz. makes it topple over. Find the weight of the bar, and the distance of its C.G. from the end.

9. If three men support a heavy uniform triangular board at its three corners, what portion of the weight will each bear?

10. A triangular board is hung by a string attached to one corner. What point in the opposite side will be in a line with the string?

11. A circular table, whose mass is 12 lb., is supported on three vertical legs, placed at equal distances on the circumference. Find the least mass which, when hung at the extremity of the diameter through one of the legs, will cause the table to topple over.

12. Having given a four-sided lamina with unequal sides, describe how you would find its C.G. (i) geometrically, (ii) experimentally.

13. A uniform triangular lamina, whose sides are 3, 4, and 5 in. respectively, is suspended by a string attached to the middle point of the longest side. Draw a figure showing clearly the position of the opposite angular point. If equal weights be attached to the three angular points, how will the position of equilibrium be affected?

14. Explain fully the circumstances under which a rectangular block, standing on a plank which is being gradually tilted, shall topple over, being prevented from sliding by a small obstacle. As an example, take the case of a block $8 \times 5 \times 5$ cub. in.

15. A cylinder whose base is a circle 1 ft. in diameter, and whose height is 3 ft., rests on a horizontal plane with its axis vertical. Find how high one edge of the base can be raised without causing the cylinder to turn over.

16. A cylinder is attached by its base to the plane base of a hemisphere, the bases of the cylinder and hemisphere having equal radii. The solid thus formed is then found to rest in any position with its hemispherical surface in contact with a smooth horizontal plane. Find the position of its C.G.

17. A regular hexagonal plate stands vertically on one side, and is divided into two parts by a vertical line bisecting that side. Will the parts stand or fall?

18. Weights of 4, 5, and 6 lb. are hung from the corners A, B, C of a weightless lamina in the form of an equilateral triangle whose side is 12 in. Where must the triangle be supported that it may rest in a horizontal position?

19. Where is the C.G. of six heavy particles, weighing respectively, 4, 5, 6, 8, 7, and 10 oz., placed in a straight line at intervals of a foot?

20. Weights of 5, 6, 9, and 7 lb. respectively, are hung from the corners A, B, C, D of a horizontal square ABCD, whose side is 27 in.

long. Find, by taking moments about two adjacent edges of the square, the point where a single force must be applied to the square to balance the effect of the forces at the corners.

21. ABCD is a square, and O is the point of intersection of the diagonals. Like parallel forces of 3, 2, 5, 4, and 12 lb. act at the points A, B, C, D, and O. Find their centre.

*22. An isosceles triangular lamina, of weight 14 lb., and whose sides are 5, 5, and 8 in. long, respectively, stands vertically with one of its equal sides on a horizontal table. Show that a weight of 12 lb., when suspended from the upper angle, will just overturn the lamina.

*23. A body of known mass is placed so as to be capable of rotation about a fixed horizontal axis; describe some method of finding the distance between the C.G. of the body and the axis.

24. On a circular table with three legs, placed at equal distances on the circumference, a weight of 96 lb. is placed so that the distances of the weight from two adjacent sides of the triangle formed by the legs are, respectively, $\frac{1}{3}$ and $\frac{1}{2}$ the altitude of the triangle. Find the force of pressure on each leg due to the weight.

CHAPTER XXVII.

BALANCES.

306. In this chapter we shall describe the various contrivances by which bodies are usually weighed. Remembering that weight is proportional to mass, we observe that the operation of weighing by balancing a body with known weights affords in every case a correct measure of the *mass* or quantity of matter in the body in pounds or grammes or other chosen units, and that the observed *weight* is independent of any local variations in the intensity of gravity.

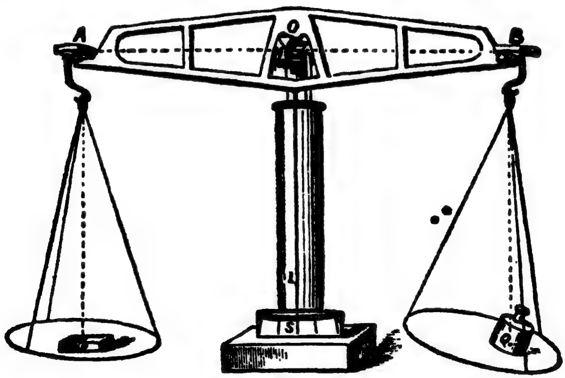


Fig. 165.

307. The common balance (see Fig. 165) consists essentially of a beam or lever **AB** fixed so that it can turn about a fulcrum **O** placed a little above its middle point. From its ends are suspended two scale pans; the goods to be weighed are placed in one of these, and are balanced by placing suitable weights in the other, till the beam assumes a horizontal position.

In delicately constructed balances, the fulcrum and points of suspension consist of wedge-shaped pieces of hard steel (called "knife blades"), whose edges rest on hard plates of steel.

The requisites of a good balance are that it be—
(i) true, (ii) stable, (iii) sensitive, (iv) rigid.

308. Conditions that the Balance may be True.—A balance is said to be true if the beam assumes a horizontal position when equal weights are placed in the two scale pans. This requires that—

(i) *The two arms of the beam must be of equal length*, that is, **AO** = **BO**, or the fulcrum **O** must be in a line **HO**, bisecting at right angles the line **AB**, which joins the points of suspension of the two scale pans.

(ii) *The scale pans must be of equal weight.*

(iii) *The* C.G. *of the beam must be vertically under the fulcrum when the beam is horizontal*, and therefore also in **HO**.

When these conditions are satisfied, equal weights placed *anywhere whatever* in the scale pans will balance each other with the beam horizontal.

For, if one of the weights, **Q**, is placed a little on one side in the scale pan instead of in the middle, the pan will swing itself a little to the other side so as to bring its C.G. vertically below the point of support **B**. The student should verify this by experiment.

309. Conditions that the Balance may be Stable.—A balance is said to be stable if the beam tends of its own accord to fall into its equilibrium position. A balance would evidently be useless for weighing if its equilibrium were *unstable* or even *neutral* (§ 303).

A balance is said to be more or less stable according to the comparative readiness or reluctance of the beam to assume its equilibrium position.

Stability is secured by placing **O**, the fulcrum of the beam, a little *above* the points **G**, **H**, at which the resultant weights of the beam and the two pans act respectively.

310. Conditions that the Balance may be Sensitive.—It is not sufficient that the beam should be horizontal when the weights in the scale pans are equal. It must also indicate when they are *unequal*, by the beam assuming a non-horizontal position. This is expressed by saying that the balance must be **sensitive** (or, as some writers call it, "sensible"). In a sensitive balance, even a small additional weight placed in one scale pan should turn the beam through a perceptible angle, and the smallness of the weight which suffices to do so affords a measure of the sensitiveness of the balance and of the degree of accuracy attainable in weighing with it.

Thus a good chemical balance will indicate differences of weight down to tenths of a milligramme.

In order to enable the smallest deflection to be observed with great accuracy, the beam carries an index or pointer **I** (Fig. 165), which moves in front of a fixed graduated scale **S**, always remaining perpendicular to **AB**.

Sensitiveness may be secured at the expense of stability by making **OH** the height of the fulcrum small, and also by lengthening the arms.

311. Rigidity.—The balance must have a beam sufficiently strong not to bend under the weights which it has to carry. To secure the greatest strength consistent with lightness, the beam is usually made in the form shown in Fig. 165.

312. Summary.—We may sum up the statements of the last four articles as follows :—

The requisites of a good balance are that it should be

(1) *true*, *i.e.* the beam should be horizontal when loaded with equal weights.

Conditions.—Equal arms, scale pans of equal weight, beam properly balanced. (§ 308.)

(2) *stable*, *i.e.* the beam should return to its equilibrium position when displaced.

Conditions.—C.G. and middle point of beam *below* knife blade. (§ 309.)

(3) *sensitive*, *i.e.* the beam sensibly deflected when weights slightly unequal.

Conditions.—Height of knife blade small, arms long. (§ 310.)

(4) *rigid*, *i.e.* beam not bent by weights. (§ 311.)

313. False Balances: Double Weighing.—A balance will evidently be false if—

(i) The arms are of unequal length.

(ii) The scale pans are of unequal weight.

(iii) The beam is improperly balanced (*i.e.* G not on OH).

Of these defects, the second and third can easily be cured. They both have the effect of inclining the beam a little out of the horizontal when the scale pans are empty. This may be rectified by filing away the scale pan or beam on the heavier side, or attaching a small weight to the scale pan on the lighter side, or (in a chemical balance) by means of a weight projecting from the index which can be turned a little on one side or the other. When the beam has thus been brought level, it only remains to correct for the inequality in the arms, which cannot be remedied.

In all such cases the true weight of a body may be found by either of the two following *methods of double weighing* :—

314. The first method is to place the body in one scale pan and balance it with suitable counterpoises (*e.g.* small shot or fine sand) placed in the opposite pan. Now remove the body and replace it by weights sufficient to balance the counterpoises, and to bring the beam to the same position as before. These weights are evidently equal to the required weight of the body, however false the balance used, for they act under exactly the same circumstances and produce exactly the same effect.

This is called the *Method of Substitution*, or *Borda's Method.*

315. The second method is less simple, but it enables us to test the trueness of the balance. The body is weighed first in one scale pan and then in the other. If the two observed weights are equal, the balance is true, and each is equal to the true weight of the body. If not, the balance is false.

We shall suppose that the arms are of unequal length, but the weights of the scale pans balance one another with the beam horizontal.

Let a, b be the lengths of the arms. Let W be the true weight of a body, P, Q the weights required to balance it when it is weighed first in one scale and then in the other. Then, by taking moments about the fulcrum, we have

$$W \times a = P \times b \quad \text{(i)},$$

and

$$W \times b = Q \times a \quad \text{(ii)}.$$

By multiplication, $W^2 ab = PQab$, or

$$W^2 = PQ.$$

Therefore

$$W = \sqrt{(PQ)} \quad \text{(iii)},$$

that is, *the true weight is the geometric mean* between the observed weights.*

Example.—The two arms of a balance are in the proportion of 9 : 10. Sugar is weighed out against $\frac{1}{2}$-lb. weight, placed first in one scale pan and then in the other. To find the total true weight of sugar.

Since the two portions of sugar balance the $\frac{1}{2}$-lb. weight in the two pans, their actual weights are $\frac{1}{2} \times \frac{9}{10}$ and $\frac{1}{2} \times \frac{10}{9}$ lb.; therefore true weight of sugar $= \frac{1}{2}(\frac{9}{10} + \frac{10}{9}) = \frac{181}{180} = 1\frac{1}{180}$ lb.

316. To compare the lengths of the two arms, we have, by (i), (ii),

$$\frac{a}{b} = \frac{P}{W} \text{ and } \frac{a}{b} = \frac{W}{Q};$$

therefore

$$\frac{a^2}{b^2} = \frac{WP}{WQ} = \frac{P}{Q}, \text{ or } \frac{a}{b} = \sqrt{\left(\frac{P}{Q}\right)},$$

giving the ratio a/b.

317. **The common or Roman steelyard** consists of a beam (AB, Fig. 166) moveable about a fulcrum or knife blade C fixed near one end B. From B is suspended the scale pan containing the body to be weighed, and a moveable weight is slid along the arm CA until the beam balances horizontally. The arm is graduated in such a way that the reading P, at which the weight rests, indicates the required weight of the body.

* $\sqrt{(PQ)}$ is the *geometric mean* between P and Q.

318. To Graduate the Common Steelyard.—Let P denote the moveable weight. First let the scale pan be empty, and let O be the position of the weight when the beam balances horizontally about C. Then the point O must be marked 0 (zero).

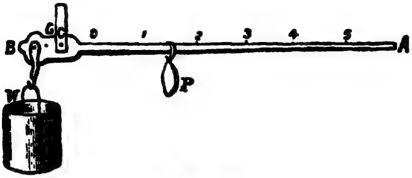


Fig. 166.

Now let a weight W be placed in the scale pan. This weight acts on the beam at B, hence its moment about C is $W \times BC$. To balance this added moment, we must increase the moment of P by an equal and opposite amount by moving it further away from the fulcrum. Thus, if P be its new position, its moment about C will be increased by $P \times CP - P \times CO$, that is, by $P \times OP$. Equating the *added* moments of P, W about C, we have therefore

$$P \times OP = W \times BC,$$

or
$$OP = \frac{W}{P} \times BC.$$

Now let I denote the position of P when the unit of weight (say 1 lb.) is placed in the scale pan. Putting $W = 1$, we have

$$OI = \frac{1}{P} BC.$$

Therefore $$OP = W . OI.$$

Hence, if $W = 2$ units, $OP = 2OI$;

if $W = 3$ units, $OP = 3OI$;

if $W = \frac{1}{2}$ unit, $OP = \frac{1}{2}OI$;

and so on. We therefore have the following rule :—

Find, by actual trial, *the points* O, I *at which* P *must be placed when the scale pan is empty and when it contains the*

unit of weight respectively. From **O** *measure off on* **OA** *successive multiples and submultiples of the length* **OI.** *Their extremities will be the points of graduation for the corresponding multiples and submultiples of the unit of weight.*

[The principle of the *Danish* steelyard is explained in Examples XXVII., Question 17.]

*319. **Roberval's Balance.**—This is a letter-balance in which the stems of the scale pans are hinged to two equal and parallel levers **AEB**, **CFD**, which turn about their middle points **E**, **F**. When the balance is slightly displaced, one of the platforms goes up and the other goes down through exactly the same distance, and the platforms always continue to remain horizontal. Hence, if equal weights **P**, **Q** are placed anywhere on the pans, the works done by them in rising and falling will be equal and opposite. Therefore, by the Principle of Work, equal weights will balance one another whatever be their positions, although one may be nearer the fulcrums **E**, **F** than the other.

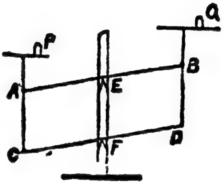


Fig. 167.

EXAMPLES XXVII.

1. How can the true weight of a body be determined by means of a balance with unequal arms?

2. The arms of a balance are 19 in. and 20 in. long respectively, and the pan in which the weights are placed is suspended from the longer arm. Find the true weight of a body which appears to weigh 14 lb. 4 oz.

3. Find the true weight of a substance which, when placed in one scale of a balance, appears to weigh 140 grm., and in the other, 154·35 grm.

4. A body whose true weight is 1¼ lb., when placed in one scale of a false balance, appears to weigh 25 oz. Find its apparent weight when placed in the other scale.

5. If the arms of a false balance are in the ratio of 24 to 25, how much per lb. does a purchaser pay for tea sold from the longer arm at 3s. per lb.?

6. If the length of the shorter arm of a balance is ·96 times the length of the other, and a body whose true weight is 15 lb. is placed in the scale pan suspended from the shorter arm, find its apparent weight.

7. The beam of a false balance is 38 in. long, and a certain body, when placed in one scale, appears to weigh 5 lb. 1 oz., and in the other, 6 lb. 4 oz. Find the true weight of the body, and the lengths of the arms of the balance.

8. The beam of a balance is a uniform straight rod whose length is 20 in. and weight 1 lb. If the fulcrum be $\frac{1}{4}$ in. on one side of the C.G. of the rod, find the true weight of a substance which apparently weighs 1 lb., the weight being placed in the pan suspended from the shorter arm.

9. A shopkeeper uses a false balance having unequal arms, whose lengths are a and b. He weighs out to two customers W pounds of coffee, as indicated by his balance; but, in serving one of the customers, he puts the weights in one scale, while, in serving the other, he puts them in the other scale. Does he gain or lose, and how much?

10. Explain how to find the true weight of a substance when the beam of the balance is quite true, but the scale pans are of unequal weight.

11. Can the steelyard be employed to determine whether or not the weight of a body is the same in different places?

12. A piece of lead placed in one pan A of a balance is counterpoised by 100 grm. in the other pan B. When the same piece of lead is placed in the pan B, it requires 104 grm. in the pan A to balance it. Find the ratio of the arms of the balance.

13. A common steelyard weighs 10 lb. The weight is suspended from a point 4 in. from the fulcrum, and the centre of gravity of the steelyard is 3 in. on the other side of the fulcrum. The moveable weight weighs 12 lb. Where should the graduation corresponding to 1 cwt. be situated?

14. In a common steelyard whose mass is 21 lb., the point of suspension of the body to be weighed is 4 in. from the fulcrum, while the centre of gravity of the beam is 1 in. from the fulcrum, measured in the opposite direction. If the mass of the sliding weight be 7 lb., find the distances from the fulcrum of the graduations marked respectively, 14 lb., 28 lb., 56 lb., and 112 lb.

15. A uniform bar 2 ft. long and weighing 3 lb. is used as a steelyard, being supported at a point 4 in. from one end. Find the greatest and least weights which can be weighed with a moveable weight of 2 lb.

16. In a common steelyard, the distance of the fulcrum from the point of support of the weight is 1 in., and the moveable weight is 6 oz. To weigh 15 lb., the moveable weight must be placed 8 in. from the fulcrum. Where must it be placed in order to weigh 16 lb.?

*17. A Danish steelyard consists of a beam AB terminating in a heavy knob at A and carrying a scale pan suspended from B. The beam hangs by a loop of string whose position can be varied, thus forming a moveable fulcrum. The weight of the beam is 1 lb., and it balances horizontally with the scale pan empty and the fulcrum distant 1 ft. from B. If now weights of 1 lb., 2 lb., and 3 lb. are successively placed in the scale, find where the fulcrum must be placed for the beam to balance. (The weight of the scale pan may be neglected.)

EXAMINATION PAPER XIV.

1. How would you find, experimentally, the C.G. of a lamina? Explain why your method holds good.

2. Explain why a man who has to carry a heavy box in one hand must throw his body on one side.

3. A uniform cylinder whose height is equal to the diameter of its base is placed on an incline of 30°; through what angle must the plane be turned so that the cylinder, which cannot slip, may be on the point of toppling over?

4. Explain why in a common scale pan or letter balance it does not matter whereabouts on the pan the weights are placed, although they may be sometimes near and sometimes further off the fulcrum.

5. What are the requisites of a good balance? How are they secured?

6. Explain a method of double weighing in a balance, and show that any inequality in the arms of the balance will not affect the accuracy of the result obtained.

7. A body, the weight of which is 2 lb., when placed in the scale of a false balance, appears to weigh 30 oz. Find its weight when placed in the other scale pan.

8. Describe the common steelyard, and show how to graduate it. What advantage is gained by the use of a steelyard?

9. A heavy uniform rod, 6 ft. long and weighing 36 lb., is kept in equilibrium, with one end against a smooth vertical wall, and the other against a smooth horizontal plane, by a rope tied to a point in the wall and the lower end of the rod. If the rope and the rod are inclined to the horizon at angles of 30° and 45° respectively, find the pressure against the wall and the tension of the rope.

[For the former equate moments about the point of contact with the ground; for the latter resolve all the forces horizontally.]

10. Five forces acting on a particle are represented in magnitude and direction by the straight lines AB, BC, CD, AF, and FE of the hexagon ABCDEF. Find their resultant.

CHAPTER XXVIII.

EXPERIMENTS.

320. This Chapter contains a series of experiments which illustrate the fundamental principles of Mechanics. It is advisable for the student to perform as many of these as circumstances permit. Those which he cannot perform should be carefully studied so as to ensure a clear understanding both of the principles at issue and of the practical details of the experiments.

321. Acceleration

Exp. 1.—*To show that the acceleration of a sphere or cylinder rolling down an inclined plane is uniform.*

The formula $s = \frac{1}{2}ft^2$ determines the distance travelled from rest under uniform acceleration. This formula shows that the distance travelled varies as the square of the time from rest. It is sufficient then to show that this relation holds in the case of the rolling sphere.

Equal intervals of time can be measured by a metronome, an instrument which can be adjusted to tick seconds, half-seconds, or any suitable intervals.

A series of experiments must then be made in which the sphere starts from rest exactly at a tick of the metronome. This is most easily done by using an iron sphere and holding it in position at the top of the plane by an electro-magnet (as described in Exp. 2). On releasing a key the electric current is stopped, the magnet ceases to attract, and the sphere is instantaneously released.

By repeated trial it is possible to adjust a ruler across the plane in such a way that the sphere when released at one tick of the metronome strikes the ruler exactly at another tick. In this way we can measure accurately the distance travelled from rest in any given number of ticks.

If the interval between successive ticks of the metronome is x seconds the results can now be tabulated as follows :—

Distance travelled in inches	3·3	13	29·5	53
Time from rest in seconds	x	$2x$	$3x$	$4x$
Distance ÷ (time)^2	$\frac{3\cdot3}{x^2}$	$\frac{3\cdot25}{x^2}$	$\frac{3\cdot28}{x^2}$	$\frac{3\cdot31}{x^2}$

Since distance ÷ (time)^2 is constant, within the limits of experimental error, we infer that the distance from rest is proportional to the square of the time, and hence that the motion is one of uniform acceleration.

Note that it is *not* necessary to know the value of x. The metronome may be set to tick at any convenient rate.

If it is required to deduce the value of the constant acceleration we require to know the value of the interval of time which the metronome beats. The metronome is set beating and coincident with one beat of the metronome a stop watch is set going. The initial beat is called 0 and the successive ones 1, 2, 3; . . . . The watch is stopped when 100 is reached and the time taken for the metronome to BEAT 100 INTERVALS is thus found. The value of the metronome interval can now be calculated.

Now from the equation $s = \frac{1}{2}ft^2$ we have $f = \frac{2s}{t^2}$. The average value of $\frac{s}{t^2}$ in the foregoing set of experiments is the average value of the last line in the table, which is $\frac{3\cdot285}{x^2}$. Hence $f = \frac{2s}{t^2} = \frac{6\cdot57}{x^2}$, where x has been determined.

NOTE.—The acceleration of a sphere rolling down an inclined plane will be *less* than that of a body sliding down the same plane if perfectly smooth (§ 153). Perfectly smooth planes are unattainable, but if a weight riding in a small trolley with approximately frictionless wheels is allowed to run down a slope a comparatively close approximation to the motion of a body down a smooth plane will be obtained. This motion can be investigated by the above method.

Exp. 2.—*To prove that the acceleration due to gravity is uniform.* (See § 46.)

MORIN'S EXPERIMENT.—A cylindrical drum is kept in uniform rotation about a vertical axis AB (Fig. 168). This uniform rotation is obtained by clockwork. A small iron ball P drops past the drum

in such a way that a small pointed pencil attached to it is constrained to touch the surface of a sheet of squared paper which is wrapped tightly round the drum. If the drum were not revolving the pencil would trace a vertical line, but owing to the motion of the drum the line traced on the paper is a curve.

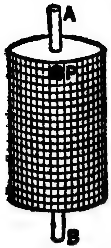


Fig. 168.

The ball P is held in position by the use of a small electro-magnet. A piece of soft iron is bent into the form of a horse-shoe M (Fig. 169), and is wound with a coil of copper wire covered with insulating material such as cotton. One terminal of the coil is connected to one of the poles of a battery C, and the other end to a key K and thence to the other pole of the cell. So long as the key K is pressed down a current circulates round the coil and the horse-shoe is magnetised, so that if one of the limbs of the magnet be placed in contact with P, P will be supported. If, however, the key K is released, the electric circuit is broken, so that the current ceases, the horse-shoe loses its magnetism, and P is released. The instant of release of P will more nearly coincide with the opening of the key K, if the poles of the electro-magnet M are furnished with a small portion of copper or other non-magnetic material.

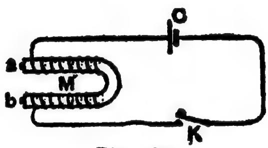


Fig. 169.

While P is supported by the electro-magnet, the drum is set in uniform rotation. The electric circuit is broken at K, and P commences to fall. It is constrained by guides to fall vertically, the constraint being as slight as circumstances permit. In falling P records its descent upon the squared paper. The paper is unwrapped and opened out. Owing to the rotation of the drum the record is not a vertical line but is found to be a curve such as *O A A′* in Fig. 170.

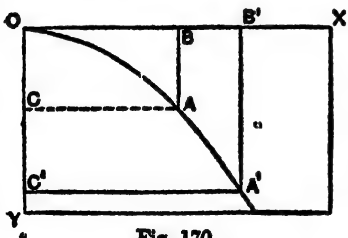


Fig. 170.

The axis OX gives us a scale of time and OY the distance P has descended. Thus if we take any point A on the curve obtained and draw AB parallel to OY and AC parallel to OX, the length OC is the distance P descends in the time represented by OB; for OB represents the amount of rotation of the cylinder and is therefore proportional to the time. Similarly, for any other point A′, OC′ is the descent in time represented by OB′.

A number of points are taken on the curve and tabulated thus :—

Time (= OB).	Distance (= OC.)	$\dfrac{\text{Distance}}{(\text{Time})^2}$.

It will be found that the third column is, within the limits of error of the experiment, a constant.

We conclude therefore that the distance descended is proportional to the square of the time of descent. This is in accord with equation $s \div t^2 = \frac{1}{2}f$ (§ 36) which holds for uniformly accelerated motion, and hence we can say that the acceleration of a freely falling body is uniform.

322. Atwood's Machine.

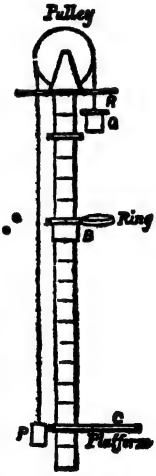


Fig. 171.

Atwood's machine, as stated in § 95, is an instrument employed for illustrating the laws of motion experimentally. A description of the machine is given in § 96. Modern improvements of the machine have made it an instrument capable of yielding reliable quantitative results in the hands of a careful experimenter.

In the modern machine, the rider **R** is of iron and **Q** and **R** are held in position by means of the electro-magnet previously described. In one form of the machine, in place of the light string joining **P** and **Q**, a paper tape is employed, the weights being attached to this by a loop. The tape is continuous, and thus no error due to the passage of the tape from one side to the other of the pulley is introduced. In this form likewise the times are recorded by the vibration of a steel strip carrying an inked brush. This brush lightly touches the tape and records its vibrations, and also enables distances of descent to be measured. As this instrument is rather too elaborate for present purposes we shall content ourselves by measuring our intervals of

time by the metronome. The value of the metronome interval is determined as in § 321, Exp. 1.

Exp. 3.—*To verify that in Atwood's machine the acceleration of the moving masses (before the rider is removed) is constant.*

It is sufficient to show that the space described from rest by Q and R in a given time is proportional to the square of the time, that is to say that $s \propto t^2$.

A rider R is placed on the mass Q, P and Q having the same mass. Q and R are retained near the top of the column by the electro-magnet. The ring B is removed. The electric circuit is broken and QR starts to descend coincident with a tick of the metronome. The platform C is adjusted by repeated trial so that QR strikes C coincident with the next tick of the metronome. In this manner the distance described by QR from rest in one metronome interval is found. QR is now placed in its initial position as before, released as before, and C adjusted so that QR strikes C coincident with the next tick but one following the release tick. In this manner the distance described by QR in 2 metronome intervals is determined. Proceeding thus we determine the space described by QR in 1, 2, 3, 4, 5, ... intervals of time. We can now tabulate our results thus :—

Interval of Time $= t$.	Space Described $= s$.	$\frac{s}{t^2}$
1	8·1 cm.	8·1
2	32·0 cm.	8·0
3	72·5 cm.	8·1
4	133·0 cm.	8·3
⋮		⋮

It will be found that the values in the third column will be approximately equal (the slight inequalities being due to experimental errors). It follows, therefore, that the space described from rest by QR is proportional to the square of the time of fall. Thus in twice the time the space described is increased fourfold, and in three times the time the space described is increased ninefold, and so on.

By equation (3) of § 36 we therefore conclude that the motion of P and QR is uniformly accelerated.

Exp. 4.—*To show that, in Atwood's machine, the velocity acquired by the masses in a given time (under the acceleration produced by the rider) is proportional to the time, i.e. $v \propto t$, and also that the square of the velocity is proportional to the distance from rest, i.e. $v^2 \propto s$.*

For this experiment the ring **B** is replaced. Note particularly that the letters t, s, v in the above statement apply to the motion before **B**, that is to the motion under constant acceleration. A suitable rider **R** is placed on **Q**, and then **QR** held in position by the electro-magnet.

(i) **QR** is released as usual coincident with a tick of the metronome and the ring **B** adjusted so that **R** is removed coincident with the next tick of the metronome. A large number of trials is made to obtain the best result. The platform **C** has now to be adjusted so that **QR** being released at a certain tick, and **R** being removed at **B** at the next tick, **Q** strikes **C** at another definite tick. The number of ticks between the passage past **B** and arrival at **C** is noted (say n). The initial distance from the start to **B** is determined (say 6·1 cm.), and the distance from **B** to **C** (say 24·4 cm.).

(ii) **B** is now adjusted so that two metronome intervals elapse between the release of **QR** and the removal at **B**, and **C** is adjusted so that the number of ticks between the passage past **B** and the arrival at **C** is again equal to n. The corresponding distances as in first case are measured—say distance from start to **B** = 24·3 cm., **BC** = 48·6 cm.

(iii) Experiments are now made so that 3, 4, 5, . . . intervals elapse between the release of **QR** and the passage past **B**, **C** being in each case adjusted so that n intervals elapse between the removal of the rider at **B** and the mass striking **C**. The corresponding distances are measured, and the results are tabulated as indicated in the table below.

It is to be observed that after removal of the rider **R** at **B**, since the mass **Q** is equal to that of **P**, the motion between **B** and **C** is uniform. The numbers in the third column, being the spaces described in equal intervals of time in the several cases, may therefore be taken as proportional to the uniform velocities in the second part of the motion, and these are *the velocities acquired during the first part of the motion.* Thus the values of **BC** are proportional to the values of v in the above formulae.

The results are tested in the fourth and fifth columns as follows:—

Number of Intervals between Release of QR and Arrival at $B \propto t$.	Initial Distance from Start to $B = s$.	Space Described in n Intervals after Rider Removed $= BC \propto v$.	$\dfrac{\text{3rd col.}}{\text{1st col.}} \propto \dfrac{v}{t}$	$\dfrac{(\text{3rd col.})^2}{\text{2nd col.}} \propto \dfrac{v^2}{s}$.
1	6·1 cm.	24·4	24·4	97·6
2	24·3 cm.	48·6	24·3	97·2
3	55·0 cm.	73·5	24·5	98·2
4	⋮	⋮	⋮	⋮
⋮				

The values in the fourth column indicate a constant, likewise those in the fifth column. If follows, therefore, that the velocity acquired from rest in a given time is proportional to the time, and the square of the velocity acquired is proportional to the space described.

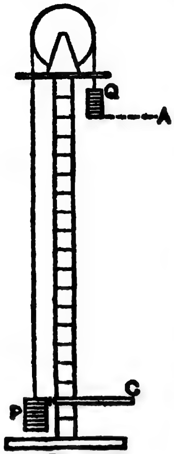


Fig. 172.

In Exps. 5 and 6 we use Atwood's machine (Fig. 172) to investigate the action of a known force (the difference of the weights of Q and P) on a known mass (the sum of the masses of Q and P).

Exp. 5.—*To show that the acceleration is proportional to the acting force, the moving mass being constant.*

In order to do this we make each of the cylinders P and Q in the form of small detachable sections. Suppose Q the heavier. The masses are arranged so that Q is in contact with the electro-magnet, the ring B is removed and C is arranged so that when Q starts falling coincident with one tick of the metronome, it strikes C coincident with a succeeding one, the number of intervals in between being noted. We have by Equation (3), § 36, $f = 2s/t^2$; hence knowing s and t we can calculate the value of the acceleration.

We now alter the value of the acting force *by transferring some of the sections from* P *to* Q, so that the total mass of P and Q remains constant. The value of the acceleration is determined as before.

We tabulate our results thus :—

Mass of system $= Q + P =$ constant ; acting force $= Q - P$.

Q	P	$Q-P$	s	t	$f=2s/t^2$	$f \div (Q-P)$
17	15	2	8·9	3	1·98	·99
18	14	4	7·9	2	3·95	·99
19	13	6	12·0	2	6·0	1·00

The values in the last column will be found (within the limits of experimental error) to be constant, so that the acceleration is proportional to the acting force.

Exp. 6.—*To show that the acceleration is inversely proportional to the mass moved, the acting force remaining constant.*

The procedure here is much the same as in Exp. 5, the cylinders P and Q consisting, as there, of a number of small detachable sections. In this case, we start with Q greater than P and determine the acceleration as in Exp. 5. Then *equal numbers* of sections are removed from Q and P so that Q − P remains constant. In this way Q + P is varied while Q − P remains constant. The acceleration in each case is determined as in Exp. 5, and we tabulate thus :—

Acting force $= Q - P =$ constant ; mass moved $= Q + P$.

Q	P	$Q+P$	s	t	$f = 2s/t^2$	$f \times (Q+P)$
18	14	32	7·9	2	3·95	126·4
17	13	30	8·5	2	4·25	127·5
16	12	28	2·3	1	4·6	128·8

It will be found that the values in the last column, *i.e.* the values of $f \times (Q + P)$ will be constant (within the limits of experimental error) and hence the value of the acceleration is inversely proportional to the moving mass, the acting force remaining constant.

Now if F is the acting force—

Experiment 5 shows that $f \propto F$ when m is constant.

Experiment 6 shows that $f \propto 1/m$ when F is constant.

We therefore have that $f \propto F/m$ when F and m both vary.

Exp. 7.—*To find the value of g.*

For this, it is necessary to determine the value of the metronome interval. Use the method described in Exp. 1, § 321.

Q and P being now given any suitable values, the value of the acceleration is determined as in Experiment 5 or 6. The acceleration is given by $f = \frac{2s}{t^2}$ where s is the distance described from rest in time t.

Now we know that $f = \frac{Q - P}{Q + P} . g$.

Knowing therefore the values of Q, P, and f we can determine g, for

$$g = \frac{Q + P}{Q - P} . f$$

$$= \frac{Q + P}{Q - P} \cdot \frac{2s}{t^2}.$$

A series of experiments, keeping Q and P fixed, must be performed, and the mean value of $\frac{2s}{t^2}$ substituted in the above formula.

Exp. 8.—*To verify that the Kinetic Energy gained is equal to the Potential Energy lost.*

In order to verify this law we require to determine the velocity of P and Q after Q has descended a certain height and P risen an equal amount.

As in Exp. 4, platforms B and C are arranged so that the mass Q being released from the position A coincident with one beat of the metronome, the ring B is reached coincident with a subsequent beat, and the platform C is reached coincident with a later beat of the metronome (Fig. 10). The distances AB and BC are measured and the time taken to describe BC ascertained. The experiment is repeated several times and the mean values of the various quantities employed are used in the calculation.

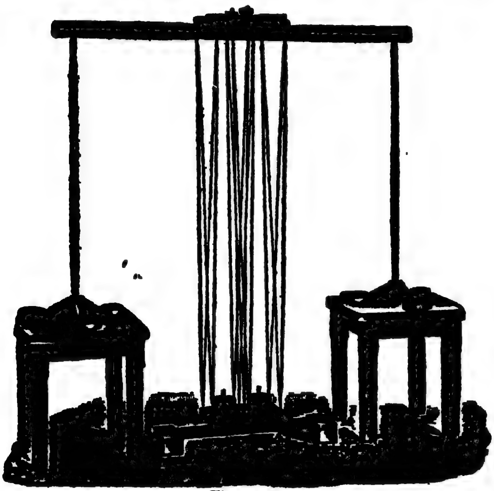


Fig. 173.

Since P ascends an amount equal to what Q descends, we have the loss of potential energy between the passage of Q from A to B = (Q − P) AB in gravitational units (ft. lb.) = (Q − P) AB . g absolute units. Also gain of kinetic energy between A and B = $\frac{1}{2}$ (Q + P)v^2, where v is the

velocity of Q on passing B : v is given by the expression $\frac{BC}{t}$, t being the time taken to describe BC and is determined by the metronome The values of $(Q-P)$ AB.g and $\frac{1}{2}(Q+P)\frac{(BC)^2}{t^2}$ are compared and within the limits of experimental error found to be equal. The experiment should be repeated for a large number of different values of Q and P and of the respective distances AB and BC.

323. Hick's Ballistic Pendulum.

Fig. 173 represents the Ballistic Pendulum made by Messrs. G. Cussons, Ltd., The Technical Works, Lower Broughton, Manchester, who have kindly supplied the illustration.

Essentially this apparatus consists of the two masses M and m, each supported by 8 threads so that each mass swings like a pendulum without twisting during its motion. Each of the masses M and m is made up of a platform upon which a series of masses can be placed. Each of the platforms is made partly of iron, so that they can be retained in any desired positions (Fig. 174) by means of electro-magnets. When the platforms are hanging freely they are just in contact.

The complete arrangement is illustrated herewith (Fig. 174). E_1 and E_2 are electro-magnets serving to keep the masses M and m withdrawn to any desired positions. The electro-magnets are in circuit with a key K. This key being opened, the masses M and m are released simultaneously. It will be found that, so long as the supporting strings are not displaced more than 10° from the vertical, the two masses meet at their lowest positions, so that both are travelling horizontally and the impact is direct: this will be the case even though the masses are unequal and the strings pulled back through unequal angles.

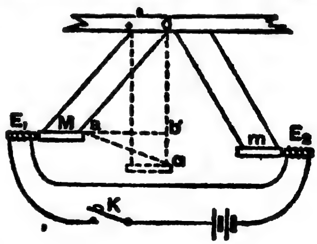


Fig. 174.

To calculate the velocity acquired by the mass M—

In passing from the initial position to the lowest position,

$$\text{loss of Potential Energy} = Mg.bc,$$

and

$$\text{gain of Kinetic Energy} = \tfrac{1}{2}MU^2.$$

Hence by the conservation of energy

$$\tfrac{1}{2}MU^2 = Mg.bc, \quad \text{i.e. } U^2 = 2g.bc \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Now let $bc = h$, $ac = l$, $ad = cd = r$.

Then

$$ad^2 - ac^2 = (ab^2 + bd^2) - (ab^2 + bc^2)$$
$$= bd^2 - bc^2.$$

Hence

$$r^2 - l^2 = (r - h)^2 - h^2,$$
$$\therefore\ l^2 = 2hr \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

Also from (ii) $\quad U^2 = 2gh.$

Dividing

$$\frac{U^2}{l^2} = \frac{g}{r},$$
$$\therefore\ U = l\sqrt{\frac{g}{r}}.$$

Now if bc is small l (or ac) is practically equal to ab. Hence in that case we have

$$U = ab\sqrt{\frac{g}{r}}.$$

The length ab is read off from the horizontal scale shown in Fig. 173.

The velocity of each body after impact is calculated in the same way from observation of the extreme position to which it swings back after the blow.

Exp. 9.—*To verify the Principle of Conservation of Momentum.*

It is now easy to verify the equation $MV + mv = MU + mu$ (§ 82), taking proper account of the signs of U, u, V, v. For M and m are determined by weighing, while U, u, V and v are calculated from the initial and final positions of M and m.

The bodies can be prevented from separating after impact by fastening two light springs at the right end of M which will grip m at the collision. This has the distinct advantage that there is now only one position to be observed after the impact—a point of the greater importance in that when the two bodies recoil separately they reach their positions of rest at the same moment.

If the experiment is arranged so that the velocity of the two masses after impact is zero, it will be easy to verify that the velocities before impact are inversely proportional to the masses.

324. The Spring Balance.

Exp. 10.—*To find how the extension of a spring depends upon the load which it supports.*

Obtain a long spiral spring. Drive a nail into a stout wooden rod and support the rod vertically in a clamp and stand (Fig. 175).

Hang a boxwood millimetre scale from the nail and then the spring in front of it. The lower end of the spring, after forming a loop, is twisted back and finally turned to form an index *p* pointing to the scale divisions; or the end is passed axially through a cork that carries a needle (*n*) horizontally (Fig. 176). Suspend a scale-pan from the lower loop by a piece of string so that it hangs clear of the support, etc.

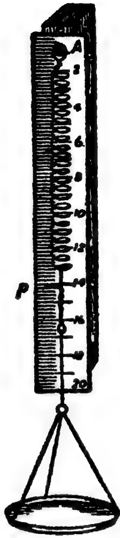


Fig. 176.

Fig. 175.

Observe the reading of the pointer, *p*, when the pan is unloaded, and when loaded successively with 20, 40, 60, etc., grammes, taking readings as the load is increased and also as it is decreased by the same steps. An error will occur in reading *p* if *p* is not almost in contact with the scale. If the spring hangs so that *p* is far from the scale, slightly rotate the spring on its vertical axis until *p* comes near the scale: then read.

Plot the readings with reference to loads. The graph is practically a straight line, showing that the extension is proportional to the load (or force) applied. From the graph deduce the extension per gramme and the load required to produce unit (*i.e.* 1 cm.) extension.

Note that the extension of the spring for any load means the difference between the reading for that load and the reading for no load.

The numbers in the third column will indicate a constant.

Otherwise tabulate as follows:—

Load.	Extension.	Extension ÷ Load.

Exp. 11.—*To graduate a spring balance.*

Remove the boxwood scale and fix, with drawing pins or small nails, a strip of paper behind the spring. Mark on the paper the position of the pointer for no load and for loads of 10, 20, 30, etc., grammes. Number the lines 0, 10, 20, etc., and subdivide the spaces between into halves or fifths.

We have thus made a scale to the balance, so that when the pointer points to any mark P we know that the load in the scale-pan is P grammes.

Exp. 12.—*Find by the spring balance of the last experiment the weights of coins, etc.*

Place the body in the scale-pan; note the number of the division to which the pointer is drawn. Check by weighing on an ordinary balance.

The term **dynamometer** simply means a force-measurer. An ordinary balance measures only a particular kind of force, viz. weights. Since, however, spring balances can be used in any position, they can be used to measure forces in general, *e.g.* if I hold a spring balance in my left hand and extend the spring with the other till the pointer reads one pound, my right hand is exerting on my left hand, and therefore also my left hand on my right hand, a force equal to the weight of one pound.

325. Conditions of Equilibrium of Forces acting at One Point.

Exp. 13.—*Equilibrium under two forces.*

Fig. 177.

Obtain a small ring **A** and tie it by strings to two spring balances **S** and **R** (Fig. 177). Fix **S** on a table, and pull **R** until both strings are quite taut. Read the balances **S** and **R**. Draw out **R** further and further. Take frequent simultaneous readings of **S** and **R**.

Observation.—In every case the readings of **S** and **R** are equal, and the two strings are in the same straight line.

Deduction.—The ring **A** is in equilibrium under the two forces exerted by the stretched strings **R** and **S**. Hence, if two forces are in equilibrium, they (i) are equal, (ii) act in the same straight line, (iii) act in opposite directions.

Exp. 14.—*Equilibrium of a particle under the action of three forces.*

Tie two spring balances **P** and **Q** to the ring **A** (Fig. 178), and fasten the balances to a table. Pull at the ring **A**: the balances at once indicate that they are exerting forces on **A**.

Tie a third balance **S** to **A**, and pull it out until **P**, **Q**, **S** are all extended. Then fasten **S** to the table. At this stage the ring is in

equilibrium under three forces, viz. the pulls of the three strings AP, AQ, and AS. Read P, Q, and S: these readings are the measures of the three forces.

Observations. — (1) However large the ring may be, the lines of the three strings, if produced, always *meet at a point* — not necessarily at the centre of the ring.

(2) No one of the forces is greater than the other two together.

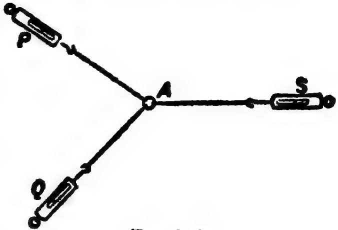


Fig. 178.

Exp. 15. — *To verify the parallelogram of forces.*

(1) Use the method described in § 168. It is advisable to use specially constructed pulleys in which the friction has been reduced to a minimum. Otherwise the pulls in the strings AH and AK may differ considerably from the weights of P and Q.

(2) Use the apparatus described in Exp. 14 above, using a similar geometrical construction to that in § 168.

Exp. 16. — *To verify the triangle and polygon of forces* (§§ 170, 173).

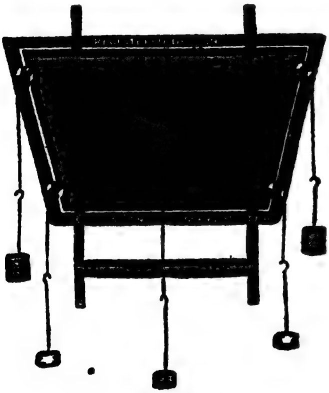


Fig. 179.

A convenient apparatus for the proof of the polygon of forces is shown in Fig. 179. This apparatus is made by Messrs. G. Cussons, Ltd., The Technical Works, Lower Broughton, Manchester, who have kindly supplied the illustration. It consists of a slotted frame and blackboard. The pulleys are very nearly frictionless and are moveable along the slots. The cords lie close against the board or paper; they are attached to a small ring shown in the middle of the figure. The frame is adapted for fixing to a wall. Slot weights are very convenient.

After having obtained the positions of the strings and the magnitudes of the forces as in Fig. 49 (p. 158), construct the polygon of forces as in Fig. 50, and show that it is a closed polygon.

Note that the directions of the lines in Figs. 49, 50 are the directions of the corresponding strings, but the lengths of the lines are not the lengths of the strings but are *proportional to the pulls* in the corresponding strings.

This experiment may also be carried out with spring balances, as Exp. 14 above.

326. Moments.

Exp. 17.—*To verify the principle of moments* (§ 201).

Bore a clean hole O (Fig. 180) through a smooth board. Place it on a smooth table and drive a smooth round nail through the hole at O. Support the board on three or four marbles so that it can turn freely round O. At any point A of the edge of the board attach a spring balance and hold it horizontally so that any force exerted by it would cause motion round O in the direction of the hands of a clock. At any other point B attach another spring balance so that a force exerted by it would cause rotation in the opposite direction.

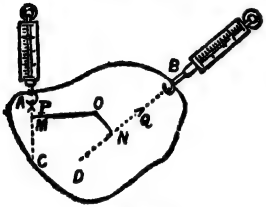


Fig. 180.

Pull the balances out so that the springs are well stretched and then fasten them to the table. Mark the directions CA DB of the pulls of the springs on the board.

Draw OM, ON perpendicular to CA and DB.

Measure OM, ON and note the readings of the balances A and B, giving the forces P and Q which they exert on the body.

Repeat the experiment several times, varying the positions of the balances. Arrange your results as in the following table :—

Values of P.	Values of OM.	Products of P and OM.	Values of Q.	Values of ON.	Products of Q and ON.

Observation.—Each result in the third column is equal to the corresponding result in the last column.

Deduction.—Since the body is at rest, we conclude that the tendency of the force P to rotate the body in one direction is exactly balanced by the tendency of Q to rotate the body in the contrary direction, *i.e.* the two tendencies are equal in magnitude, that is, their moments

about O are equal; and hence we infer that the moment of a force should be measured by the product of the force and the perpendicular from the centre of rotation on its line of action.

COR. The moment of a force P about a point O is zero if
either (i) the force P is zero,
or (ii) the line OM is zero, in which case O lies on the line of action of P.

Conversely, If the moment of a force P about O is zero, it follows that either (i) P is zero,
or (ii) O lies on the line of action of P.

327. Equilibrium of Three Parallel Forces.

Exp. 18.—*To verify the laws for the equilibrium of three parallel forces.*

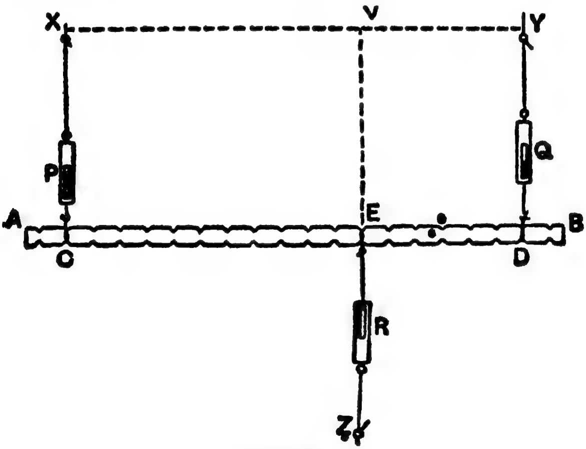


Fig. 181.

Use the method of § 218, or modify it as follows :—

Apparatus.—(i) Obtain a rod of wood, about 1 m. long, 2 cm. wide, and 1 cm. thick. Graduate the bar in centimetres. At intervals of 5 cm. cut notches into the rod. Prepare little loops of string to slide along the rod (Fig. 181).

(ii) Obtain three spring balances. Tie long pieces of string to the rings at the top of the balances.

Adjustments.—Pass the hooks of two of the balances P and Q through loops placed at C and D, C and D being near the ends. Measure CD carefully, and measure an exactly equal distance XY near an edge of the experimenting table (Fig. 8). At X and Y insert two strong picture rings firmly in the table. Pass the strings of the balances P and Q through the rings at X and Y. Since CD = XY, it is possible to arrange the length of the strings and balances so that XY and CD are parallel and the angles at C and D are right angles. When this is done, fasten the strings to the rings, leaving the string at X and Y so that they can be loosened or tightened if required.

Pass the hook of the third balance R through a third loop at E on the wooden rod. Note the distance DE. In the straight line XY measure YV = DE; in the line VE produced, mark any point Z, and insert a third strong picture ring in the table at that point. Through this ring pass the string of the balance R until it becomes just taut. In this position there is no tension on the balances.

Procedure.—(i) Pull out the balance R by drawing the string further through Z. Immediately all three balances indicate tension; but the system may become distorted, in which case the reading of the balances tells us nothing about *parallel* forces.

(ii) A little adjustment of the strings, probably a tightening at Y, will, however, restore the parallelism.

We now have three parallel forces in equilibrium, viz. the three tensions of the spring balances, which act on the rod at the points C, D, E, but produce no motion in the rod. Hence either of these three tensions may be regarded as equal and opposite to the resultant of the other two.

Measurements.—Note the readings of the balances P, Q, R, and the distances CD, CE, DE.

When several sets of readings have been obtained in this way, test whether they satisfy the equations

$$P + Q = R, \text{ and } \frac{P}{DE} = \frac{Q}{EC} = \frac{R}{CD}.$$

328. Resultant of Two Parallel Forces (§§ 210, 212).—The rules for the resultant of two parallel forces can be verified from the preceding experiment.

Thus the resultant of the two *like* parallel forces P and Q is equal and opposite to R. It is therefore equal to $P + Q$.

Also its line of action divides CD internally in the ratio $CE : DE$, which from the second equation is equal to the ratio $Q : P$.

Similarly the resultant of the two *unlike* parallel forces P and R is equal and opposite to Q. It is therefore equal to $R - P$.

Also its line of action divides CE externally in the ratio $CD : ED$, which by the second equation is equal to the ratio $R : P$.

329. Loaded Beams (§ 221).

The apparatus shown in Fig. 182 (manufactured by Messrs. Townson & Mercer, Ltd., 34, Camomile Street, London, E.C., who have kindly supplied the illustration) can be used to illustrate the conditions of equilibrium for loaded beams.

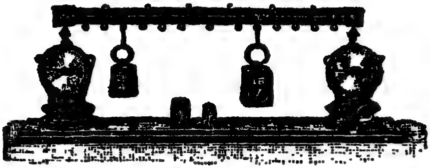


Fig. 182.

These conditions are—

(1) That the sum of the upward forces on the beam is equal to the sum of the downward forces (the upward forces are the forces registered by the two compression balances; the downward forces are the weight of the beam and the suspended weights).

(2) That the sum of the clockwise moments about any point on the beam is equal to the sum of the anticlockwise moments.

330. Centre of Gravity.

Exp. 19.—*To determine the centre of gravity of a body practically.*

This problem has already been considered in §§ 297, 298.

The centre of gravity of a rod can also be determined by placing it on a table with a portion projecting over the edge. Increase the projecting portion till the rod is on the point of falling over. The point of the rod now at the edge of the table is the centre of gravity.

The centre of gravity of a sheet of cardboard can be determined by the same method. When the cardboard is on the point of falling over, mark with a pencil the line on the cardboard which coincides with the edge of the table. This line contains the centre of gravity. Repeat the experiment with the cardboard in a new position: the intersection of the two lines is the centre of gravity.

331. Couples.

Exp. 20.—*To show that, if two couples acting on a body in the same plane balance, their moments are equal and opposite* (cf. § 227).

Work on a bench having a smooth and even surface. Apparatus required : 4 clamps, 4 spring balances, 4 nails, string, a piece of wood, some peas or marbles.

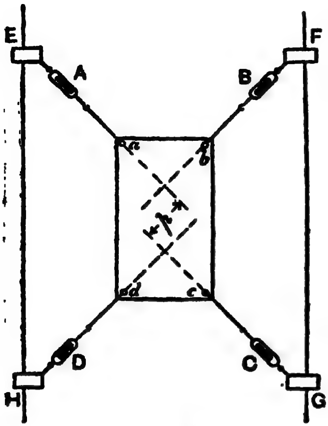


Fig. 183.

Drive 4 nails *a*, *b*, *c*, *d* (Fig. 183) into the piece of wood near the corners. Tie string to the nails. Place some peas or marbles on the bench and rest the wood on them. The peas or marbles give the wood an easy freedom of motion over the bench.

Take the clamps and fasten two to each side of the bench. Tie a spring balance to each of them. Take the string from each nail and loop it over the hook of the nearest balance. Arrange so that all the four balances are in tension, and then adjust the clamps and lengths of the strings until the readings of A and C are the same and A*a* and C*c* are parallel. It will then be found that the readings of B and D are the same and that B*b* and D*d* are parallel.*

Measure the distances p between A*a* and C*c* and q between B*b* and D*d*. Show now that

the reading of A (or C)$\times p =$ the reading of B (or D)$\times q$,

i.e. the moments of the two couples under which the wood is in equilibrium are equal and opposite. Repeat the experiment with the clamps in different positions.

332. Levers (Ch. XXII., §§ 232-240).

Exp. 21.—*To make a simple lever.*

A simple form of lever can be made from a half-metre scale graduated in millimetres. To prepare it, bore clean circular holes about a millimetre or two millimetres in diameter at every 5 cm. division along the scale (Fig. 184). It is convenient to bore them not along the central line but nearer the upper edge. To make a fulcrum, drive a stout sewing needle or a thin smooth wire nail into the edge of the bench, or a board fixed in the wall, or a piece of wood which can be held in a strong clamp. Other apparatus required for the experiment includes some

* A couple can be balanced by another couple or by a system of *more than two* forces equivalent to a couple. It cannot be balanced by two forces which do *not* form a couple.

weights, say ounce weights and fractions of a pound, making up to two pounds, provided with either rings or suitable projections for the purpose of suspension by strings; a spring balance reading in ounces up to two pounds or more; and some cotton or fine string.

Exp. 22.—*To deduce the principle of the lever when the weights are hung on either side of the fulcrum.*

Balance the lever (made as above described) with its central hole C (Fig. 184) supported by the needle. If it does not balance level tie a piece of fine wire tightly round the lighter end, adjusting the length or position of the wire until it does balance level.

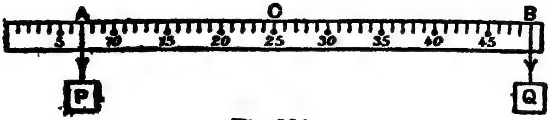


Fig. 184.

Take two weights P and Q (say $P = 4$ ounces, $Q = 3$ ounces), and suspend them by loops of cotton from points A and B on the lever, one on each side of C, so that P and Q tend to turn the lever in opposite directions. Place P, say, 18 cm. from C, then find exactly where Q must be placed to balance, *i.e.* to make the lever set horizontal; you will find that Q must be placed at 24 cm. from C.

Observe that $4 \times 18 = 3 \times 24,$

i.e. $P \times \mathrm{AC} = Q \times \mathrm{BC}.$

Repeat the experiment with P and Q at different distances from C. The following distances may be tried for AC:—

$$\mathrm{AC} = 9 \text{ cm.}, 13\tfrac{1}{2} \text{ cm.}, 18 \text{ cm.}, 27 \text{ cm.},$$

and you will find that

$$\mathrm{BC} = 12 \text{ cm.}, 18 \text{ cm.}, 24 \text{ cm.}, 36 \text{ cm. respectively.}$$

Now repeat the experiment with different weights, say $P = 8$ oz., and $Q = 1$ lb., $P = 6$ oz., $Q = 9$ oz., etc.

In each case you will find that when P and Q balance

$$P \times \mathrm{AC} = Q \times \mathrm{BC}.$$

We may call AC the arm of P, and BC the arm of Q; our relation then becomes:—

The force on one side of the fulcrum multiplied by its arm is equal to the force on the other side of the fulcrum multiplied by its arm.

This is the *Principle of the Lever.*

N.B.—We have here neglected to take account of the weight of the scale, because the two sides balance each other.

Exp. 23.—*Without using a balance to find the weight of a uniform lever.*

Take the half-metre scale of Exp. 21, hang it with the fulcrum F at the hole on the 5 cm. mark (Fig. 185). Hang a weight P on the short arm, and shift it along the lever until the lever balances horizontally under the combined action of P and the weight of the lever. Suppose

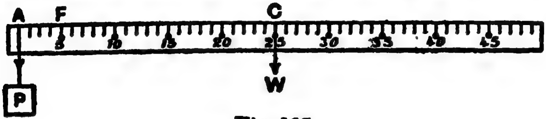


Fig. 185.

P then hangs at A. The weight W of the scale acts downwards at its middle point C. Hence, applying the Principle of the Lever, we get

$$P \times \mathrm{AF} = W \times \mathrm{FC}.$$

$$\therefore\ W = P \times \frac{\mathrm{AF}}{\mathrm{FC}}.$$

In the case under consideration $\mathrm{FC} = 20$ cm. Suppose that $P =$ 10 oz. and $\mathrm{AF} = 4$ cm., then

$$10 \times 4 = W \times 20.$$

$$\therefore\ W = 2 \text{ oz.}$$

Repeat the experiment, using the hole at the 10 cm. mark for the the fulcrum. You may find that if $P = 8$ oz., $\mathrm{AF} = 3{\cdot}7$ cm., whence

$$8 \times 3{\cdot}7 = W \times 15,$$

or
$$W = \frac{29{\cdot}6}{15} = 2 \text{ oz. approx.}$$

Repeat the experiment, using in turn the 15 cm. hole, the 20 cm. hole, etc., for the fulcrum.

Verify the result by weighing the half-metre scale on an ordinary balance or a spring balance.

Exp. 24.—*To make a simple lever of the second system.*

Take the half-metre scale of Exp. 21. We may adopt one of two plans :—

(1) Balance the scale on its middle point, in which case we do not take account of the weight of the scale.

(2) Use for a fulcrum a hole near the end of the scale, in which case we have to allow for the weight of the scale.

Exp. 25.—*Using the lever of* **Exp. 24 (1),** *to find the law of a lever of the second system.*

Take the half-metre scale and hang it with its middle point C on the fulcrum (Fig. 186). Slip two loops of cotton over one end. Hang a convenient weight *R*, say, on the inner loop and place the loop at a point A, say, on the lever. To the other loop attach a spring balance, pulling upwards, and lift the balance until the balance and string are vertical and the lever horizontal. If the balance does not record an exact number of ounces, shift it along the lever until it does. Read the balance and its distance from C, viz. BC. Let *E* be the force exerted by the balance. Show by your observations that the moments about C are equal, *i.e.*

$$R \times CA = E \times CB.$$

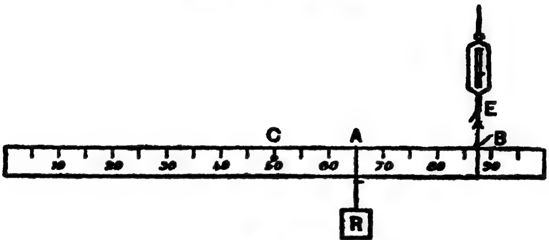


Fig. 186.

Repeat the experiment, using different values of *R*, of CA, and of BC, and show that all your results lead to the same conclusion.

Exp. 26.—*Using the lever of* **Exp. 24** (2), *to find its weight without using an ordinary balance.*

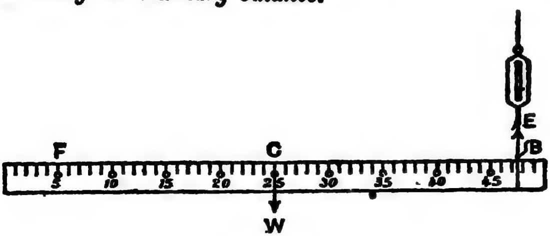


Fig. 187.

Place the fulcrum F at the 5 cm. hole (Fig. 187) and support the long end by a spring balance which is capable of reading accurately to a much smaller weight than that of the lever. When the lever is horizontal, take the reading of the balance, *E*.

Then, by the principle proved above, we get

$$E \times \mathrm{BF} = W \times \mathrm{FC} = W \times 20,$$

$$\therefore\ W = \frac{E \times \mathrm{BF}}{20}.$$

Exp. 27.—*To verify the principle of the lever with a lever of the third class.*

Modify either of Exps. 25, 26, placing the spring balance closer to the fulcrum than the weight it supports.

333. Inclined Plane (see §§ 187, 188).

It is useless to try to obtain a *smooth* plane, that is to say one along which a body will *slide* without resistance. The best results are obtained by making the plane as *even* as possible and using a body which is free to *roll* up or down the plane.

Exp. 28.—*Inclined plane with force acting up the plane.*

Obtain two pieces of smooth board, AB, AC (Fig. 188), about 24 in. long and 4 in. wide. Hinge them together at the end A. Clamp AC to the bench or table. If a small block of wood, D, be inserted between the boards, AB may be inclined at any desired angle. We thus obtain an inclined plane which is easily adjusted to any slope.

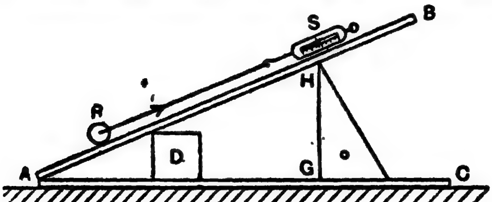


Fig. 188.

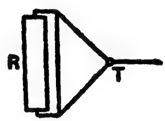


Fig. 189.

Obtain a small, fairly heavy, smooth metal cylinder R mounted smoothly on a framework provided with a hook T (see Fig. 189), to which can be attached a string.

Test the cylinder and plane for friction by placing R on AB and slightly tilting AB. If the friction is small, as it should be, the cylinder will begin to move almost as soon as AB is tilted at all. If the plane AB is not very smooth the difficulty may be got over by resting a piece of glass on it.

Weigh the cylinder and the attached framework and adjust the plane to any convenient slope. Tie a piece of cotton to T and to a spring

balance, place the cylinder on the plane, and hold the spring balance so that the string lies parallel to the slope: the spring balance will register the force P which, acting parallel to the plane, is necessary to support the weight W of the cylinder and framework.

It will be found that, owing to friction, the force required to keep the body at rest will vary between certain limits. To get a definite value, find first the least force required to make the cylinder move upwards, then the least force that will just prevent it moving downwards. The mean of these two may be taken as the equilibrating force, *i.e.* $P = \frac{1}{2}(P_1 + P_2)$.

In order to measure the height, length, and base of the plane, place a large set square underneath as shown. HG gives accurately the height of the plane for a length AH and base AG; AH and AG can easily be measured by a millimetre scale and HG can be measured once for all.

In this experiment measure AH, HG. Repeat the experiment many times with different inclinations for AB. Tabulate the results thus:—

GH	AH	P	W	GH/AH	P/W

Observation.—It will be found that the corresponding values in the fifth and sixth columns are equal.

Deduction.—When the supporting force is *parallel to the plane* the following law holds:—

$$\frac{\text{supporting force}}{\text{weight}} = \frac{P}{W} = \frac{\text{GH}}{\text{AH}} = \frac{\text{height of plane}}{\text{length of plane}}$$

Exp. 29.—*Inclined plane with horizontal force.*

Repeat the last experiment, keeping the string *horizontal*. The board AB must be narrow enough to pass through the triangle in the framework of Fig. 189 without touching it. Measure GH, AG and tabulate the results thus:—

AG	GH	W	P	GH/AG	P/W

Observation.—The value in the fifth column is always equal to the corresponding value in the sixth.

Deduction.—When the supporting force is *horizontal*,

$$\frac{\text{supporting force}}{\text{weight}} = \frac{P}{W} = \frac{\text{GH}}{\text{AG}} = \frac{\text{height of plane}}{\text{base of plane}}.$$

The *pressure on the plane* in the last two experiments can be found by determining the force, applied at right angles to the plane, which is just sufficient to lift the roller off the plane. This can be done by means of a spring balance joined by string to each end of the axle of the roller.

334. Forces in a Crane (§ 191, Ex. 1).

Exp. 30.—*To illustrate the forces in the tie and jib of a simple crane when loaded.*

The apparatus required is shown in Fig. 190. The upright post **K** is known as the king-post, **TO** is the tie-bar, and **JO** is the jib. When the crane is loaded the tie-bar is put into tension and the jib in compression. A *light* spring balance **A** is inserted in the tie and a compression balance **B** (like an ordinary kitchen balance) in the jib. The tie can be fastened to any of the different eyes fixed to the king-post. As the compression balance is to measure the thrust along the jib, there must *not* be a hinge between the jib and the balance. If the jib is to point in different directions it and the balance must swing together.

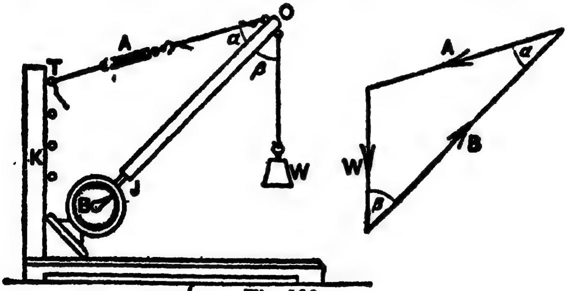


Fig. 190.

For any given position and a given weight W read the balances and measure the angles α and β. Draw the triangle of forces (see second figure), using the known values of W and α and β. From the triangle, deduce the tension in the tie and the compression in the jib and compare the values obtained with those observed at A and B.

APPENDIX.

PROPERTIES OF CERTAIN TRIANGLES. RESULTS IN MENSURATION.

1. The student who is unacquainted with Trigonometry will find it necessary to become familiar with the relations between the sides and angles of certain right-angled triangles which are constantly occurring in problems in Mechanics.

2. The units of angular measure are the subdivisions of a right angle, defined as follows :—

1 right angle = 90 degrees, denoted by 90° ;

1 degree or 1° = 60 minutes, denoted by 60′ ;

1 minute or 1′ = 60 seconds, denoted by 60″.

The only angles we shall have to consider are 30°, 45°, 60°, 90°, and their multiples.

3. **Relation between the sides of a right-angled triangle.**

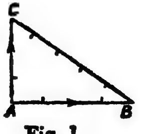


Fig. 1.

By Euc. I. 47,

$$BA^2 + AC^2 = BC^2,$$

where $\angle BAC = 90°$.

4. **The 3, 4, 5 and 5, 12, 13 Triangles, etc.**

Thus *the numbers* 3, 4, 5, or their multiples (*e.g.* 6, 8, 10), *are proportional to the sides of a right-angled triangle, for* $9 + 16 = 25$, *i.e.* $3^2 + 4^2 = 5^2$. These numbers should be remembered. Another such set is 5, 12, 13, for $5^2 = 25 \times 1 = (13 + 12)(13 - 12) = 13^2 - 12^2$, and $\therefore 5^2 + 12^2 = 13^2$.

5. The Right-Angled Isosceles Triangle (Fig. 2).

THEOREM.—If the angles of a triangle be 45°, 45°, and 90°, the sides are proportional to 1, 1, and $\sqrt{2}$.

Draw a square ABCD, and draw the diagonal AC.

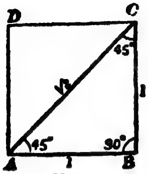


Fig. 2.

Then $\angle BAC =$ half a right angle $= 45°$,
$\angle BCA =$ half a right angle $= 45°$,
$\angle CBA =$ a right angle $= 90°$;

$\therefore \quad AC^2 = AB^2 + BC^2$. (Euc. I. 47.)

Also $\quad AB = BC$;

$\therefore \quad AC^2 = 2AB^2 = 2BC^2$;

$\therefore \quad AC = \sqrt{2}.AB = \sqrt{2}.BC$;

$$\therefore \quad \frac{BC}{AC} = \frac{1}{\sqrt{2}}, \ \frac{AB}{AC} = \frac{1}{\sqrt{2}}, \ \frac{BC}{AB} = 1;$$

or

$$\frac{AB}{1} = \frac{BC}{1} = \frac{AC}{\sqrt{2}}.$$

6. The Semi-Equilateral Triangle (Fig. 3).

THEOREM.—If the angles of a triangle be 30°, 60°, and 90°, the sides opposite these angles are proportional to 1, $\sqrt{3}$, and 2.

Draw an equilateral triangle ABC. Join A to D, the middle point of BC. Then the triangles ABD, ACD are equal in every respect.

But the three angles of an equilateral triangle are all equal, and are together = two right angles = 180°; therefore each = 60°.

$\therefore \quad \angle DBA = 60°$;

$\therefore \quad \angle DAB = 30°$.

Also $\quad \angle ADB = \angle ADC = 90°$;

$\therefore \quad AB^2 = AD^2 + DB^2$. (Euc. I. 47.)

But $\quad AB = CB = 2DB$;

$\therefore \quad 4DB^2 = AD^2 + DB^2 \quad$ or $\quad AD^2 = 3DB^2$;

$\therefore \quad AD = \sqrt{3}.DB$;

$$\therefore \quad \frac{DB}{AB} = \frac{1}{2}, \ \frac{AD}{AB} = \frac{\sqrt{3}}{2}, \ \frac{DB}{AD} = \frac{1}{\sqrt{3}};$$

or

$$\frac{BD}{1} = \frac{DA}{\sqrt{3}} = \frac{BA}{2}.$$

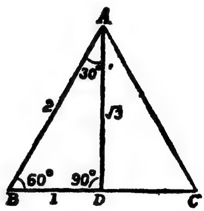


Fig. 3.

7. The student is recommended to remember the figures and properties of the right-angled isosceles triangle and the semi-equilateral triangle, as the solution of almost every *simple* problem in Mechanics involving angles can be deduced from them. In drawing the figures from memory, it is useful to remember that the greater angle is opposite the greater

side, and vice versa. Thus, in the semi-equilateral triangle, since 2 or $\sqrt{4} > \sqrt{3} > 1$, the side represented by 2 is opposite the angle 90°, the side represented by $\sqrt{3}$ is opposite the angle 60°, and so on.

8. **Similar Triangles.**—If two triangles have the angles of one respectively equal to the angles of the other, then one of the triangles will be an exact copy of the other on an enlarged or reduced scale, so that the sides of one triangle will be proportional to those of the other. Such triangles are said to be **similar**. (Euc. VI. 4.)

Thus, if the triangles ABC, DEF (Fig. 4) have

angle at A = angle at D,

angle at B = angle at E,

and $\therefore$ angle at C = angle at F, (Euc. I. 32)

their sides will be proportional, so that

$$\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}.$$

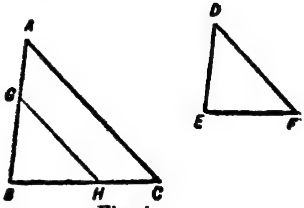


Fig. 4.

In particular, if GH be drawn parallel to AC, the triangles ABC, GBH will be similar, so that

$$\frac{BH}{BC} = \frac{HG}{CA} = \frac{GB}{AB}.$$

9. Several converse theorems are also true. If the sides of one triangle are proportional to those of another, the angles of one triangle will be equal to those of the other, and the triangles will therefore be similar. (Euc. VI. 5.)

Again, if AB : BC = DE : EF, and the included angles at B, E are equal, the triangles will be similar. (Euc. VI. 6.)

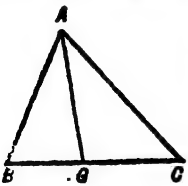


Fig. 5.

10. If a line AG be drawn bisecting the angle at A of a triangle ABC (Fig. 5), the segments of the base BC will be proportional to the adjacent sides (Euc. VI. 3), so that

$$\frac{BG}{GC} = \frac{BA}{AC}.$$

The triangles ABG, ACG are not, however, *similar*.

11. **Results in Mensuration.**

The following facts in Solid Geometry and Mensuration are assumed. The references given below are to the articles in Briggs and Edmondson's *Mensuration*, where the reader will find the properties in question fully

proved. Proofs of them are also given in most elementary treatises on Solid Geometry. The *results* alone need be remembered :—

(1) The area of a triangle

$$= \frac{1}{2}\,(base) \times (altitude). \qquad (\S\ 45.)$$

(2) The area of a trapezoid (*i.e.* a quadrilateral with two sides parallel)

$$= (its\ height) \times (\tfrac{1}{2}\ sum\ of\ parallel\ sides). \qquad (\S\ 49.)$$

(3) The length of the circumference of a circle of radius r

$$= \pi \times (diameter)$$

$$= 2\pi r\,; \qquad (\S\ 57.)$$

where the Greek letter π ("pi") stands for a certain "incommensurable" number (that is, a number which cannot be expressed as an exact arithmetical fraction), whose value lies between 3·141592 and 3·141593. The following approximate values should be remembered and used, unless otherwise stated:—

$$\pi = \frac{22}{7}, \text{ for all } rough \text{ calculations;}$$

$$\pi = 3{\cdot}1416, \text{ more approximately.}$$

(4) The area of the circle

$$= \frac{1}{2}\,(radius) \times (circumference)$$

$$= \pi r^2. \qquad (\S\ 58.)$$

(5) The volume of a pyramid

$$= \frac{1}{3}\,(height) \times (area\ of\ base)$$

$$= \frac{1}{3}\,hA, \qquad (\S\ 106.)$$

the height h being the perpendicular from the vertex on the plane of the base, and A the area of the base.

(6) The area of the curved surface of a cylinder, whose height is h and the radius of whose base is r,

$$= (height) \times (circumference\ of\ base)$$

$$= 2\pi rh. \qquad (\S\ 115.)$$

(7) The volume of the cylinder

$$= (height) \times (area\ of\ base)$$

$$= \pi r^2 h. \qquad (\S\ 116.)$$

(8) The area of the curved surface of a right circular cone, whose height is h and the radius of whose base is r,

$$= \frac{1}{2}\,(\textit{circumference of base}) \times (\textit{length of slant side})$$

$$= \pi r \sqrt{(h^2 + r^2)}; \qquad (\S\ 117.)$$

a *slant side* being a line drawn from the vertex to a point in the circumference of the base.

(9) The volume of the cone

$$= \frac{1}{3}\,(\textit{vol. of cylinder of same base and height})$$

$$= \frac{1}{3}\pi r^2 h. \qquad (\S\ 118.)$$

(10) The area of the surface of a sphere of radius r

$$= 4 \textit{ times area of circle of same radius}$$

$$= 4\pi r^2. \qquad (\S\ 126.)$$

(11) The volume of the sphere

$$= \frac{1}{3}\,(\textit{radius}) \times (\textit{surface})$$

$$= \frac{4}{3}\pi r^3. \qquad (\S\S\ 127,\ 128.)$$

ANSWERS.

DYNAMICS.

Examples I. (Pages 16, 17.)

1. (i) 132. (ii) 1. (iii) $\frac{11}{15}$. 2. (i) 880. (ii) 400. (iii) 2.
3. (i) $3\frac{3}{4}$. (ii) $\frac{3}{8}$. (iii) 360. 4. 100 ft. 5. 1,100 ft.
6. 12 yd. 7. 660 ft. 8. 16 sec. 9. $1\frac{7}{8}$ miles per hour.
10. $26\frac{2}{3}$. 11. $17\frac{1}{7}$ ft. per sec.
12. (i) 6 ft. per sec. (ii) 94 ft. per sec. 13. 5 hours.
14. 15 sec. 15. $\frac{1}{20}u$. 16. (i) 3. (ii) $416\frac{2}{3}$.
17. (i) 1·08. (ii) 30. 18. 25 metres.
19. 10 metres. 20. $15mq : 22np$.

Examples II. (Pages 21-23.)

1. (i) Diminished to $\frac{1}{3}$ its former value. (ii) Increased to 3,600 times its former value.
2. (i) 38,400. (ii) $78,545\frac{5}{11}$. 3. $\frac{2}{3}$. 4. 225.
5. $5\frac{13}{15}$ ft. per sec. per sec. 6. 16 sec. 7. 56 ft. per sec.
8. 5 ft. per sec. per sec. 9. 40 ft. per sec.
10. 6 ft. per sec. per sec. 11. $\frac{1}{3}$ ft. per sec. per sec.
12. 450 ft. 13. $-\frac{1}{15}$ ft. per sec. per sec.
14. (i) 36, 35,280. (ii) 129·6, 127,008. 15. $\frac{11}{15}$.
16. 33 cm. per sec. per sec. 17. 5 sec.

Examination Paper I. (Page 23.)

1. See §§ 13, 24. 2. See § 27. 3. 440. 4. 5 : 11.
5. $17\frac{3}{5}$ ft. per sec. 6. See § 9. 7. $16\frac{1}{2}$.
8. 2 ft. per sec. per sec. 9. 35·1 nearly. 10. 30 miles an hour.

Examples III. (Pages 32, 33.)

1. 2,000 ft. 2. 40 miles per hour; $\frac{4}{5}$ mile.
3. 4 ft. per sec. per sec. 4. 84 ft. per sec. 5. 4 sec.

6. 8 ft. per sec. per sec. 7. 320 ft. 8. 16 ft.

9. $\frac{1}{4}$ sec., $\frac{1}{4}(\sqrt{2}-1)$ sec., $\frac{1}{4}(\sqrt{3}-\sqrt{2})$ sec. 10. $10\sqrt{10}$ sec.

11. 1,000 ft. per sec. 12. 36 ft. per sec. per sec. 13. $22\frac{1}{2}$ ft.

14. § 42 ; $2\frac{1}{12}$ miles. 15. 235 ft.

16. 22 ft. per sec. ; 6·8 ft. per sec. per sec.

17. 50 cm. per sec., 150 cm. per sec. 18. 3 ft. per sec. per sec.

19. $f : F = v^2 - u^2 : V^2 - U^2$; $t : T = V + U : v + u$. 20. Yes.

21. 10 sec.

EXAMPLES IV. (PAGES 48-52.)

1. (i) 400 ft., 160 ft. per sec. ; 12,250 cm., 4,900 cm. per sec.
 (ii) 57,600 ft., 1,920 ft. per sec. ; 1,764,000 cm., 58,800 cm. per sec.
 (iii) 51,840,000 ft., 57,600 ft. per sec. ; 1,587,600,000 cm., 1,764,000 cm. per sec.
 (iv) ·16 ft., 3·2 ft. per sec. ; 4·9 cm., 98 cm. per sec.

2. (i) 160 ft. per sec. ; 5 sec. (ii) 240 ft. per sec. ; $7\frac{1}{2}$ sec.
 (iii) 4 ft. per sec. ; $\frac{1}{8}$ sec. (iv) 1,400 cm. per sec. ; $1\frac{3}{7}$ sec.

3. 48 ft., 176 ft., 208 ft. 4. 32 ft.

5. 144 ft. ; 36 ft. below top of cliff. 6. 100 ft., 1,200 ft. per sec.

7. 156 ft. 8. 1,156 ft. 9. 16 ft. per sec., 4 sec. later.

11. 64 ft. per sec. 12. $6\frac{1}{2}$ sec. 13. 10 ft. per sec.

14. $\sqrt{gh}$. 15. 196 ft. ; 112 ft. per sec. ; 4 ft.

16. 32 ft. per sec. 17. 8 sec. 18. 240 ft. per sec.

19. 184 ft. 20. 400 ft. ; 10 sec. 21. $\frac{5}{16}$ ft.

22. $(\sqrt{3}+1)\sqrt{\frac{h}{g}}$. 23. $1\frac{1}{2}$ sec. ; 84 ft. from ground.

24. 1 or 4 sec. 25. $\frac{1}{4}$ sec. 26. 4 sec. 27. $80\sqrt{2}$ ft. per sec.

28. 4,080 ft. 29. 68 ft. per sec. nearly ; 307 ft. nearly.

30. First stone remains in contact with his hand ; second stone seems to fall exactly as it would if the cage were at rest.

31. ·45 ; 1·19 ; −·21 ; −6·2. 32. $11\frac{1}{2}°$; $52\frac{1}{2}°$; $-63\frac{1}{2}°$; $-40\frac{1}{2}°$.

33. 36 miles per hour. 34. $11\frac{1}{4}$ miles per hour.

35. 16 ft./sec. ; − 32 ft./sec. 36. 400 yd./min ; 800 yd./min.

37. Acc. = 32 ft. per sec. per sec. 38. ·098 ft. per sec. per sec.

39. 3·2 and ·5 ft. per sec. per sec. 40. 6, − 2, −8 ft. pe sec. per sec.

EXAMINATION PAPER II. (PAGE 52.)

1. See §§ 39, 40. 2. See §§ 36, 37. 3. $\frac{4}{5}$ ft. per sec. per sec.
4. 75 ft.; 30 ft. per sec. 5. See §§ 41, 42. 6. $2\frac{1}{4}$ ft. per sec. per sec.
7. See §§ 43, 47. 8. See § 56. 9. 196 ft.; 48 ft. per sec.
10. 2 sec. after the second stone was thrown down, 5 ft.

EXAMPLES V., VI. (PAGES 62–64.)

1. (i) 20 F.P.S. units. (ii) 107,520 F.P.S. units.
   (iii) 28,800 F.P.S. units. (iv) 98,100 C.G.S. units.
2. (i) $32\sqrt{10}$ F.P.S. units. (ii) 30 F.P.S. units.
   (iii) 8,960 F.P.S. units. (iv) 1·4 C.G.S. units.
3. 60 F.P.S. units. 4. 6,720 F.P.S. units. 5. 144 poundals.
6. $9{\cdot}97\dot{3}$ sec. 7. 350 poundals. 8. $6\frac{1}{4}$ poundals.
9. 64 poundals. 10. 10 sec.; 9 ft. 11. $nP : mQ$.
12. 96 poundals.
13. 14,336,000 poundals; 1,433,600 F.P.S. units. 14. $1\frac{3}{5}$ lb.
15. 1,971,200 poundals, 440 ft. 16. $\frac{1}{2}\frac{P}{m}n^2$ ft., $\frac{1}{2}\frac{P}{m}(2n-1)$ ft.
17. 128 poundals. 18. 3 sec.
19. 100 F.P.S. units of impulse.
20. 1,500,000 F.P.S. units of impulse.
21. 1,700,000 F.P.S. units of impulse. 22. 4·5 cm.
23. 50,000 dynes. 24. 64·8 metres. 25. $\frac{3}{10}$. 26. 9,800 dynes.
27. $27{\cdot}\dot{7}$ dynes. 28. $\frac{1}{12}$ poundal; 5 F.P.S. units.
29. 25,940·25 dynes. 30. 10 tons wt., $13\frac{1}{3}$ tons wt.

EXAMINATION PAPER III. (PAGES 64, 65.)

1. See § 61. 2. See § 65; 3,200 F.P.S. units.
3. See § 68. 4. 8 poundals.
5. (i) $\frac{1}{70}$ ft. per sec. per sec.; 960 F.P.S. units.
   (ii) 50 cm. per sec. per sec.; 1,500 C.G.S. units.
6. (a) $65{,}706\frac{2}{3}$ poundals. (b) $10{,}951\frac{1}{9}$ poundals.
7. $7\frac{1}{2}$ lb. weight. 8. 1 poundal. 9. 654 cm. per sec.
10. $\frac{1}{8144}$ lb. weight.

Examples VII. (Pages 71, 72.)

1. 2 ft. per sec. 2. 10 ft. per sec. 3. 40 cm. per sec.
4. $\frac{1}{4}$ ft. per sec. 5. 10 ft. per sec.; 2 ft. 6. 896,000 poundals.
7. $\frac{1000}{25601}$ ft. per sec. 8. 30 ft. per sec. 10. 3 : 1.
11. 30 ft. per sec., opposite to the motion of the larger piece.
12. $2\frac{1}{3}$ ft. per sec. 13. 20 cm. per sec.
14. 6 ft. per sec. 15. 4 ft. per sec.

Examples VIII. (Pages 79-81.)

1. (i) 4 ft. per sec. per sec.; 240 ft. per sec.; 7,200 ft.
   (ii) 640 ft. per sec. per sec.; 38,400 ft. per sec.; 1,152,000 ft.
   (iii) ·981 cm. per sec. per sec.; 58·86 cm. per sec.; 1,765·8 cm.
2. 25 ft. 3. 8 sec. 4. 3200 F.P.S. unit.
5. 10 sec.; 36 ft. 6. 1,962 cm. 7. 5 : 1.
8. $12\frac{4}{5}$ oz. 9. 11 : 1,280. 10. 4,905 $\frac{nl^2}{m}$ metres.
11. Almost exactly 14 cm. per sec.
12. $6\frac{1}{4}$ lb. weight; $\frac{5}{38}$ ft. per sec. per sec.
13. (i) 7 lb. 11 oz. weight; 237 lb. 6 oz. weight. 14. $\frac{1}{80}$ sec.
15. 15 poundals. 16. 31 ft. per sec. per sec.
17. $\frac{2}{3}$ ton weight; $\frac{1}{3}$ ton weight. 18. Weight of 12 oz.
19. 225 lb. weight; 175 lb. weight. 20. 34 lb. weight.
21. 605 ft. 22. 32·18 ft. per sec. per sec.
23. 6,250 lb. weight. 24. 63·5688 cm.

Examination Paper IV. (Page 81.)

1. See §§ 77, 79. 2. 2 ft. per sec. 3. 14 ft. per sec.
4. See §§ 5, 64, 86. 5. See § 70. 6. 3,200 F.P.S. units.
7. 325 lb. weight. 8. See §§ 91, 93.
9. $1\frac{7}{8}$ lb. weight. 10. $4\frac{11}{28}$ tons weight.

Examples IX. (Pages 90-92.)

1. (i) 2 ft. per sec. per sec.; $15\frac{15}{16}$ lb. weight $31\frac{7}{8}$ lb. weight.
   (ii) 24 ft. per sec. per sec.; weight of $3\frac{1}{2}$ oz.; weight of 7 oz.
   (iii) 16 ft. per sec. per sec.; 168 lb. weight; 336 lb. weight.
   (iv) 8 ft. per sec. per sec.; $11\frac{1}{4}$ lb. weight; $22\frac{1}{2}$ lb. weight.
   (v) $10\frac{2}{3}$ ft. per sec. per sec.; 4 lb. weight; 8 lb. weight.

(vi) 1 cm. per sec. per sec.; 490$\frac{490}{981}$ gm. weight; 980$\frac{980}{981}$ gm. weight.
(vii) 639 cm. per sec. per sec.; 31$\frac{41}{109}$ gm. weight; 62$\frac{82}{109}$ gm. weight.
(viii) 109 cm. per sec. per sec.; $4\cdot\dot{4}$ kilog. weight; $8\cdot\dot{8}$ kilog. weight.

2. (i) 17 ft. per sec. per sec.; 15 ft. per sec. per sec.; 7$\frac{31}{32}$ lb. weight.
(ii) 28 ft. per sec. per sec.; 4 ft. per sec. per sec.; weight of 1$\frac{3}{4}$ oz.
(iii) 24 ft. per sec. per sec.; 8 ft. per sec. per sec.; 84 lb. weight.
(iv) 20 ft. per sec. per sec.; 12 ft. per sec. per sec.; 5$\frac{5}{8}$ lb. weight.
(v) 21$\frac{1}{3}$ ft. per sec. per sec.; 10$\frac{2}{3}$ ft. per sec. per sec.; 2 lb. weight.
(vi) 491 cm. per sec. per sec.; 490 cm. per sec. per sec.; 245$\frac{245}{981}$ gm. weight.
(vii) 810 cm. per sec. per sec.; 171 cm. per sec. per sec.; 15$\frac{75}{109}$ gm. weight.
(viii) 545 cm. per sec. per sec.; 436 cm. per sec. per sec.; weight of $2\cdot\dot{2}$ kilog.

3. 384 ft. 4. 1$\frac{4}{5}$ lb. 5. $\frac{1}{7}$ lb.
6. 3 ft. per sec.; 4$\frac{1}{2}$ ft. 7. 18 ft.; weight of 15$\frac{3}{4}$ oz.
8. 8 sec. 10. 8 ft. 11. 30 lb.
12. 8 ft. per sec. per sec.; 3 poundals. 13. $\frac{4}{53}$ lb.
14. 492 gm. 15. 654 cm. 16. 6 poundals; 32 ft. per sec.; 64 ft.
17. 501$\frac{6}{13}$ lb., 498$\frac{7}{13}$ lb. 18. 10 lb., 14 lb.

Examination Paper V. (Pages 92, 93.)

1. See §§ 96, 103. 2. See § 98. 3. $\frac{32}{5}$ ft. per sec. per sec.; 80 ft.
4. 15 lb. wt. 5. See § 101, and use masses 8$\frac{1}{4}$ and 7$\frac{3}{4}$ oz.
6. See § 101. 7. $\frac{1}{2}$ lb. weight; 6$\frac{3}{4}$ oz. weight, 5$\frac{1}{4}$ oz. weight.
8. 8 ft. per sec. per sec., 3$\frac{3}{4}$ lb. weight. 9. 18 lb.
10. 500$\frac{500}{981}$ gm., 499$\frac{481}{981}$ gm.

Examples X. (Pages 105-107.)

1. (i) 1,200 ft.-poundals. (ii) 53$\frac{1}{3}$ ft.-poundals.
(iii) 10,000 ergs. (iv) 10 gramme-centimetres.
2. (i) 160 ft.-poundals. (ii) 375 ft.-lb.
(iii) 8,000 ergs. (iv) 1,000,000 ergs.
3. 298$\frac{2}{3}$ ft.-poundals.
4. (i) 32,000 ft.-poundals. (ii) 36 ft.-tons.
(iii) 100 gramme-centimetres. (iv) 10 kilogrammetres.
5. 187,500 ft.-lb.

6. (i) Momenta are equal.
(ii) Kinetic energy of M : kinetic energy of $m = m : M$.
7. 12,600 ft.-lb. 120 ft. per sec.
8. 1,010 lb. weight.
9. 18·26 ft. per sec. nearly; 219·1 F.P.S. units nearly; 2,000 ft. poundals.
10. The shot can do 162 times as much work as the gun.
11. See §§ 65, 109, 114.
12. 480 F.P.S. units: 144 ft.-lb.
13. (i) 48,020,000 ergs. (ii) 12,005,000 ergs. (iii) 0.
14. $1\frac{41}{84}$ ft.
15. (*a*) 21,000 lb. weight. (*b*) 364·4 ft. per sec. (*c*) 155,610·1875 f[illegible]ns.
16. 12,960 ft.-poundals
17. 125 ft.-poundals.
18. 2,000 lb. weight; $\frac{1}{40}$ sec.
19. The velocity is doubled.
20. 36 miles an hour.
21. 3·2.
22. 1 hour 40 minutes.
23. 240.
24. $10\frac{5}{16}$ tons.
25. 30 miles an hour.
26. 735·75 watts.
27. ·0025 ft.-poundals; ·0000416 F.P.S. units of power.
28. 10,000,000 ergs.

Examination Paper VI. (Pages 107, 108.)

1. See §§ 106, 109, 122.
2. 554,400 ft.-lb.
3. 250 ft.-poundals.
4. See § 113.
5. $37\frac{29}{33}$.
6. 24.
7. 960 ft.-poundals.
8. Weight of 325 lb. $8\frac{1}{2}$ oz.
9. 55 lb. weight.
10. See § 115.

Examples XI. (Pages 119-121.)

1. 5 ft. per sec.; 150 ft.
2. 220 yd.
3. In a south-westerly direction; apparent vel. $= \sqrt{2}$ times actual vel.
4. Vertically downwards; actual vel. of rain $= \sqrt{3}$ times vel. of carriage.
5. Direction of steering makes an angle whose cosine is $\frac{3}{5}$ up-stream with a line drawn directly across the stream.
6. 10 miles an hour from the north-west.
7. 22 ft. to the side of the object opposite to that towards which motion is taking place.
8. $24\sqrt{2}$ miles an hour due south-west.
9. $20\sqrt{3}$ ft. per sec.
10. 9 ft. per sec.
11. 5 ft. per sec., $\sqrt{29}$ ft. per sec.
12. About 30°; $13\frac{1}{3}$ ft. per sec.
13. 13 ft. per sec.
14. (i) 4 ft. per sec.; 0. (ii) $\frac{15}{2}\sqrt{3}$ miles an hour; $7\frac{1}{2}$ miles an hour. (iii) $6\sqrt{2}$ yd. per min.; $6\sqrt{2}$ yd. per min.

(iv) 5 metres per min.; $5\sqrt{3}$ metres per min.

(v) 0; 24 ft. per sec. (vi) -5 kilom. per hour; $5\sqrt{3}$ kilom. per hour.

(vii) -4 miles an hour; 4 miles an hour.

(viii) $-6\sqrt{3}$ ft. per sec.; 6 ft. per sec. (ix) -10 cm. per sec; 0.

15. 5 miles an hour. 16. Just over 50 ft. per sec.

17. 13 ft. per sec., making an angle whose tangent is $\frac{5}{12}$ west towards the north.

Examples XII. (Pages 130, 131.)

1. 2 ft. per sec. per sec., south-east.
2. $12\sqrt{3}$ ft. per sec., at right angles to the original direction.
3. $\sqrt{2}.v$, bisecting the interior angle of the square at the corner; $mv\sqrt{2}$ units of impulse.
4. $32\sqrt{2}$ ft. per sec.; $32\sqrt{5}$ ft. per sec.; $32\sqrt{10}$ ft. per sec.

6. 350 ft. 7. No change. 8. 1,500 ft. 9. $2\frac{1}{2}$ sec., $146\frac{3}{8}$ ft.

10. 150 ft. per sec. 11. Apparent path is vertically downwards.

12. 500 ft. 13. $26\frac{19}{32}$ lb. weight. 14. 20 ft.

15. 1 sec. 16. $15,625\sqrt{3}$ ft. 18. $100\sqrt{3}$ ft. per sec.

Examination Paper VII. (Page 131, 132.)

1. See § 129. 2. 2 ft. per sec.; 30° west of south. 3. 1,100 yd. per min.

4. Straight across, 10 min. 5. $44\sqrt{3}$ ft. 6. See § 138.

7. $5\sqrt{3}$ ft. per min. if from P to Q; 5 ft. per min. if from Q to P.

8. See § 143. 10. 256 ft.; 120 ft.

Examples XIII. (Pages 136, 137.)

1. (i) 26 lb. (ii) 29 oz. (iii) 17 gm. (iv) 221 tons.

2. 30 lb. and zero. 3. 2 ft. per sec. per sec.

4. 13 lb. weight. 5. 256 ft. per sec., nearly. 6. 66 ft.

Examples XIV. (Pages 144-146.)

1. (i) 1 ft.; 2 ft. per sec. (ii) $6\frac{2}{13}$ ft.; $12\frac{4}{13}$ ft. per sec.
(iii) $\frac{3}{4}$ ft.; $1\frac{1}{2}$ ft. per sec.

2. (i) 1,280 ft.; 256 ft. per sec. (ii) 448 ft.; 89·6 ft. per sec.
(iii) $752\frac{11}{17}$ ft.; $150\frac{9}{17}$ ft. per sec.

3. 16, $16\sqrt{2}$, and $16\sqrt{3}$ ft. per sec. 4. 21·1 miles an hour, nearly.

5. $\sqrt{30}$ sec.; $16\sqrt{30}$ ft. per sec. 6. $112\frac{1}{2}$ ft., $3\frac{3}{4}$ sec.

7. $65\frac{5}{48}$ ft., $5\frac{5}{24}$ sec. 8. 56 ft. 9. $16\sqrt{3}$ ft. per sec.; $\frac{3}{2}\sqrt{3}$ sec.; $\frac{1}{2}\sqrt{3}$ sec.

10. One-eighth as far up the second plane as up the first.
11. 107·96 lb. weight, 1,870 ft.-lb. 12. $5\frac{1}{3}$ tons. 13. 900 ft.-lb.
14. 247·68. 15. $23\frac{4}{7}$ ft. per sec. 16. 490·5 cm. per sec.; 245·25 cm.
17. (i) $\frac{50}{109}$ cm. (ii) 5·5 cm. (iii) 50 cm. 18. $49{,}050\sqrt{3}$ ergs.
19. 490·5 cm. per sec. per sec.; 30°. 20. 3·2 ft. per sec. per sec; 40 ft.
21. 10 lb. 22. 3 : 2. 23. 4 ft. per sec. per sec. 24. 4 ft. per sec

Examination Paper VIII. (Pages 146, 147.)

1. See § 150. 2. 17 lb. 3. The forces are in equilibrium.
4. See § 153. 5. 8 ft. 6. 112 ft. 7. $24\frac{3}{8}$ lb.
8. P and Q are equal. 9. See § 154, Cor. 1. 10. $\frac{1}{8}$ kilogrammetre.

STATICS.

Examples XV. (Pages 162, 163.)

1. About $11\frac{1}{3}$ lb. wt.
2. Tension of lowest part = 3 lb. wt.; of middle part = 8 lb. wt.; of top part = 15 lb. wt.
3. 10 lb. and 8 lb. 4. 12 lb. and 8 lb. 5. 18 lb.; 12 lb.
6. (i) 9AP, where P divides BC so that 2PB = 7PC.
   (ii) 3PA, where P divides BC so that 2PB = PC.
   (iii) 2PA, where CB is produced to P, and 3PB = PC.
   (iv) 3PA, where BC is produced to P, and PB = 4PC.
   (v) 5AP, where BC is produced to P, and 7PC = 2PB.
   (vi) 3AP, where P divides BC so that PB = 2PC.
8. 10 lb., acting in the direction of the fourth force.
9. 4CP, where P lies in AB and AP : BP = 16 : 9.
10. (i) 25 lb. (ii) 58 lb. (iii) 169 lb. (iv) 260 lb.
11. 45 lb. 12. 30 lb. and 72 lb. 13. 48 lb. and 14 lb.
14. 136 lb. 15. $144\frac{1}{5}$ lb., $143\frac{3}{5}$ lb. 16. 3AD, in direction AD.
18. DA, where D is the middle point of BC. 19. 2AD parallel to AD.
20. 65 lb. wt. 21. Yes.

EXAMPLES XVI. (PAGES 172-174.)

1. (i) 5 lb., 0. (ii) $4\sqrt{3}$ lb., 4 lb.
(iii) 10 tons, 10 tons. (iv) 6 oz., $6\sqrt{3}$ oz.
(v) 0, 5 cwt. (vi) $-$ 16 kilog., $16\sqrt{3}$ kilog.
(vii) $-$ $2\sqrt{2}$ tons, $2\sqrt{2}$ tons. (viii) $-$ $12\sqrt{3}$ gm., 12 gm.
(ix) $-$ 3 mgm., 0.

2. $10\sqrt{5}$ lb. wt., making an angle with the vertical whose tangent is $\frac{1}{2}$.

3. $4\sqrt{3}$ lb., at 30° to the vertical. 4. 10 lb.; $5\sqrt{3}$ lb.

5. (i) 4 lb. each. (ii) 2 lb. each.

6. (i) 18 oz. (ii) $\sqrt{13}$ tons. (iii) $\sqrt{5}$ lb.
(iv) $3\sqrt{7}$ gm. (v) 145 mgr. (vi) $12\sqrt{3}$ kilog.
(vii) $\sqrt{34}$ lb. (viii) $\sqrt{113-56\sqrt{3}}$ cwt. (ix) 1 lb.

7. $\sqrt{19} : \sqrt{13} : \sqrt{7}$. 8. $\sqrt{229}$ lb.

9. $14\frac{1}{2}$ lb. along the line of action of the 12-lb. force, but in the opposite direction, and $\frac{3}{2}\sqrt{3}$ lb. perpendicular to it on the side opposite to the 5-lb. force.

11. $10\sqrt{10}$ lb. 13. $2\sqrt{58}$ lb.

14. $\sqrt{3}$ lb., acting perpendicular to the 2-lb. force, between the 2-lb. and 3-lb. forces. 15. 9 lb. and 12 lb.

16 13 lb. 17. 8 lb., $2\frac{2}{3}$ lb. 18. 60°. 20. $\frac{3}{5}$. 21. $-\frac{4}{5}$.

22. The components along the internal and external bisectors are $6\sqrt{6}$ lb. and $6\sqrt{2}$ lb., the components along the given lines are $6(\sqrt{3}-1)$ lb. and $6(\sqrt{3}+1)$ lb.

23 29·5 lb.; 23·19 lb. 24. 5·54 lb.; 3·65 lb.

26. $2\sqrt{2}$ lb., bisecting the angle COD.

27. $\sqrt{3}$ lb., acting perpendicular to the 4-lb. force between the 4-lb. and 5-lb. forces. 28. $\frac{1}{3}\sqrt{3}$ ton.

EXAMINATION PAPER IX. (PAGE 174.)

1. See § 165. 2. See §§ 162, 167. 3. See § 168. 4. See § 170.
5. 25 lb. 6. See § 173. 7. 1 lb., acting parallel to CB.
8. See §§ 179, 181. 9. See § 183. 10. 23·43 lb.

EXAMPLES XVII. (PAGES 182-184.)

1. (i) 9 lb. (ii) $17\frac{1}{2}$ lb. (iii) 33 kilog.
2. (i) 72 ft.-lb. (ii) 420 ft.-lb. (iii) 3,960 kilogrammetres.
3. (i) 45 lb. (ii) $11\frac{1}{3}$ tons. (iii) 26 gm.

4. (i) 585 ft.-lb. (ii) 1,680 ft.-tons. (iii) 2,210 gm.-cm.

5. (i) $12\sqrt{3}$ oz., 18 oz. (ii) $12\sqrt{2}$ lb., 12 lb.
(iii) $120\sqrt{3}$ gm., $60\sqrt{3}$ gm.

6. (i) $24\sqrt{3}$ oz., $18\sqrt{3}$ oz. (ii) 24 lb., 12 lb. (iii) 240 gm., 60 gm.

7. (i) 27 ft.-lb. (ii) 180 ft.-lb. (iii) $4{,}500\sqrt{3}$ gm.-cm.

8. (*a*) $64\frac{1}{2}$ lb., nearly. (*b*) 56 lb. 9. 60°. 11. 72 lb.

13. $13\frac{3}{8}$ lb., $36\frac{1}{3}$ lb. 14. $\frac{1}{3}\sqrt{3}$ kilog. wt.

15. $5\sqrt{2}$ lb. wt. each. 16. $8\sqrt{3}$ lb. wt., 8 lb. wt.

18. $\frac{1}{2}$ ton wt., or 1 ton wt., according as the string makes an angle of 60° with the upward or downward drawn vertical.

19. $20\sqrt{3}$, or 34·64 lb. wt.; 40 lb. wt.

20. $28\sqrt{2}$ lb. wt.; tension of string = or > weight of picture, according as the angle between the two parts of the cord = or > 120°.

21. Pull on cord is diminished by increasing length of cord, and increased by diminishing it.

22. 120°. 23. $\frac{40}{3}\sqrt{3}$ lb. wt.

24. $2\sqrt{3}$ lb., or $4\sqrt{3}$ lb., according as the force is applied away from or towards the plane. 25. $\frac{3}{2}\sqrt{2}$ lb.; $\frac{5}{2}\sqrt{2}$ lb.

26. 0 or $16\sqrt{3}$ lb., according as the force is exerted away from or towards the plane. 27. $10\sqrt{3}$ lb.; $20\sqrt{3}$ lb.

28. $\frac{4}{5}$ and $\frac{3}{5}$ of a ton.

Examples XVIII. (Pages 192, 193.)

4. 8 lb. 5. 225 lb.; 135 lb. 6. $10\sqrt{2}$ lb.; 10 lb.

7. 30° with the vertical; 40 lb. 8. 30° with the vertical; 9 in.

9. 27 lb., 65·8 lb. nearly.

10. $6\sqrt{3}$ lb. on bottom of box, 6 lb. on lower side.

11. 20 lb., 15 lb 12. $6\sqrt{2}$ lb. 13. $2\sqrt{7}$ lb., $2\sqrt{3}$ lb.

14. $\sqrt{3}$ lb., 1 lb 16. $\frac{1}{2}\sqrt{3}$ lb., $\frac{1}{2}$ lb. 17. 20 lb., 32·2 lb.

18. Middle point of rod is vertically under the peg; 7·2 lb., 9·6 lb.

19. 260 lb.

Examination Paper X. (Page 194.)

1. See § 188. 2. 35 lb. 3. 12 lb.; 21 lb. 4. See § 194.

5. See § 195. 6. 45 lb.; 60 lb. 7. $6\sqrt{2}$ lb.; 12 lb.

8. 125 lb. 9. $\frac{2}{3}\sqrt{3}W$. 10. 62·45 lb.

EXAMPLES XIX. (PAGES 201, 202.)

1. (i) 20, (ii) 12, (iii) 2,688 in ft.-lb. units, (iv) 50 in cm.-gm. units.

2. (i) 15, (ii) $15\sqrt{3}$, (iii) 30, (iv) $15\sqrt{2}$, (v) 15 in ft.-lb. units.

3. Moments about left-hand extremity are 0, 24, − 24, − 60; algebraic sum = − 60. Moments about right-hand extremity are 20, −96, 16, 0; sum = − 60. Moments about centre are 10, − 36, − 4, − 30; sum = − 60.

4. $5\sqrt{3}$, $10\sqrt{3}$, $20\sqrt{3}$ in ft.-in. units. **5.** 3AC, 2AC, 4AC.

6. Towards the side of OC on which B lies. **7.** 7 in.

9. $2\sqrt{21}$ in., $\sqrt{21}$ in. **10.** Resultant passes through O.

11. 0, $20\sqrt{3}$, $40\sqrt{3}$, $40\sqrt{3}$, $20\sqrt{3}$, 0 in ft.-in. units.

12. $\frac{Q}{\sqrt{P^2+Q^2}}$ AB, $\frac{Q-P}{\sqrt{P^2+Q^2}}$ AB, $\frac{P}{\sqrt{P^2+Q^2}}$ AB.

13. $\frac{9}{2}\sqrt{3}$, $9\sqrt{3}$ in ft.-lb. units. **14.** 4 in. from C in BC produced.

15. $3P$.AB, $3P$.AB, $2P$.AB, $2P$.AB.

16. The point of application of the 6-lb. force. **17.** $\frac{3}{5}\sqrt{5}$.AB.

EXAMPLES XX. (PAGES 212-214.)

1. (i) 5 lb.; 18 in.,* 12 in. (ii) 1 lb.; 6 ft., 2 ft.
(iii) 9 tons; 40 yd., 8 yd. (iv) 14 lb.; 44 ft., 12 ft.
(v) 20 gm.; 144 cm., 36 cm. (vi) 180 kilog.; 25 m., 20 m.

2. (i) 1 lb.; 90 in., 60 in. (ii) 8 oz.; 12 ft., 4 ft.
(iii) 6 tons; 60 yd., 12 yd. (iv) 8 lb.; 77 ft., 21 ft.
(v) 12 gm.; 240 cm., 60 cm. (vi) 20 kilog.; 225 m., 180 m.

3. (i) 12 lb., 4 ft. from the smaller force.
(ii) 4 lb., at the other end of the bar.

4. 25 lb.; 3 ft. **5.** 15 in. from the end on which the boy sits.

6. The resultant acts $7\frac{1}{2}$ ft. from the larger force, on the side remote from the smaller.

7. 7·2 cm. from the smaller force.

8. The pressure increases as the distance between the hand and the shoulder diminishes.

9. $2\sqrt{2}P$, acting parallel to CA at a distance $\frac{3}{4}$AC from it on the same side as D.

10. 18 in. from the end at which the larger weight is placed.

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* In Examples 1 and 2 the distance of the resultant from the smaller force is given first.

11. 96 lb. 12. $2\frac{1}{3}$ cwt., $1\frac{1}{2}$ cwt. 13. 4 lb., 3 lb. 14. 10 lb., 9 lb.
16. $(P^2 - Q^2)/P$. 17. 1 in. nearer the 4-lb. force. 18. 2 lb.
19. 60 lb. 20. 48 lb., 72 lb. 21. 2 lb., 10 in. from the table.

Examples XXI. (Pages 221-223.)

1. 10 in. from the point of application of the 2-lb. force; 20 lb.
2. $8\frac{1}{3}$ in. from the point of suspension of the 16-lb. weight; 40 lb.
3. 7 in. from the other prop.
4. 225 lb. downwards, 5 lb. upwards; 50 lb. and 170 lb., both downwards.
5. The point of attachment of the 8-lb. weight.
6. $7\frac{3}{5}$ in. from the 3-lb. force.
7. 16 in. from the end at which the 1-lb. weight is suspended.
8. 0, 380 lb., $3\frac{1}{3}$ ft. from prop. 9. 58 lb., 26 lb.
10. (*a*) $36\frac{2}{5}$ lb., $54\frac{3}{5}$ lb. (*b*) $78\frac{2}{5}$ lb., $12\frac{3}{5}$ lb.; $-5\frac{3}{5}$ lb., $96\frac{2}{5}$ lb.
11. 5 ft. from the heavier boy. 12. 3 ft. from the centre.
13. 28 lb.
14. 5 lb. acting parallel to the given force of 5 lb., and at a distance of 2 ft. from it.
15. The resultant of a force P and a couple of moment M is an equal and parallel force P at a distance M/P to the right of the original force.
17. (i) $(P + Q)q - Qp$. (ii) $(P + Q)q + Qp$.
18. 12 lb. parallel to AC, at a distance $\frac{2}{3}$AB from it, on the same side as D.
19. $2P \times$ area ABC. 20. $Pp - Qq$. 21. Along AD.

Examination Paper XI. (Page 223.)

1. See §§ 201, 203. 2. 40, 30, 24 in ft.-lb. units.
3. See § 212. 4. 180 lb., 120 lb. 5. See § 215.
6. See § 218. 7. See §§ 223, 224. 8. See § 228.
9. See § 227. 10. Moment = twice area of hexagon.

Examples XXII. (Pages 236-238.)

1. $4\frac{1}{2}$ ft., $10\frac{1}{2}$ ft. 2. 4 lb. 3. 12 lb., 48 lb.
4. 8 ft. from the smaller weight. 5. 6 in., 20 in.
6. 14 lb.; $1\frac{1}{3}$ ft. from the fulcrum. 7. 6 lb.
8. 1 ton; 2,400 lb. 9. 9 lb.

10. (i) 50 lb., 25 in.; 40 lb., $7\frac{1}{2}$ in. (ii) 62 lb., 31 in.; 72 lb., $13\frac{1}{2}$ in.
11. 8 lb. 12. 5 lb. 13. 30° with the lever.
14. 8 lb., 4 lb.; 1 : 2. 15. $\sqrt{3} : 1$. 16. 16 : 9.
17. 8 lb., 2 lb. 18. $1\frac{1}{3}$ in. 19. 360 lb.
20. 10 lb.; $1,047\frac{13}{21}$ ft.-lb. 21. 60 lb.
22. 40 in., 5 in. 23. $4\frac{1}{2}$ lb. 24. 6 tons.
25. The sum of the weights of the man and the wheel and axle.

Examples XXIII. (Pages 256-260.)

1. 60 lb. 2. 45 lb. 3. 1 : 16. 4. 31 lb.
5. 7. 6. 1 : 15. 7. 5 lb. 8. $\frac{31}{32}W$.
9. Light in the first system, heavy in the third. 10. 22 lb.
11. 16. 12. $14P$. 13. 611 lb. 14. 12 lb.
15. 120 lb. 16. 72 stone. 17. 9 stone 4 lb.; 80 lb.
18. $15P$. 19. $14\frac{4}{15}P$. 21. 32 ft.
22. In the third system, with five moveable pulleys or two single-string systems with mechanical advantages 7 and 9, the string round the pulleys of one being attached to the block of the other.
23. The first system, with three moveable pulleys, or the single-string system, with eight strings at the lower block; 24 ft.
24. 290 lb. 25. 7 : 5,280. 26. $5\frac{5}{48}$ lb. 27. $4\frac{94}{99}$ lb.
28. 2,200 ft.-lb. 29. 5 lb. 30. $1\frac{16}{33}$ lb.
31. Step $= \frac{33}{112}$ in., pitch $= \frac{3}{84}$ in.
32. $62\frac{3}{8}$ lb. 33. 64 : 1. 35. 246 lb. 36. 175 ft.
37. 1809·6 lb. 38. $1,440\pi : 1$. 39. ·61 approx.
40. 676 ft.-lb. 41. 87 per cent. 42. 15·3 h.p.
43. $\dfrac{Wv}{Pu}$; $\dfrac{pu}{100v}$.

Examination Paper XII. (Page 260.)

1. See §§ 237, 239, 240, 246. 2. $\frac{3}{7}$ ft. towards centre.
3. As a moveable pulley with parallel strings. 4. 200 lb.
5. See § 259. 6. $92\frac{1}{2}$ lb. 7. See § 266. 8. See § 262.
9. 3 lb.; 7. 10. Step $= \frac{122}{343}$ in.

Examples XXIV. (Pages 268, 269.)

1. (i) The intersection of the medians.
(ii) The point D, where ABCD is a parallelogram

2. The middle point of the line through C bisecting AB
3. 1 cwt. 4. 45 lb.
5. $\frac{1}{4}$ AB from CD along the line bisecting AB, CD.
6. The middle point of the line joining the centre of the hexagon to the corner opposite the one at which the unlike force acts.
8. 4 lb. 9. 7 in. from the centre of the 7-lb. sphere.
10. $2\frac{1}{4}$ ft. from the 2-lb. wt. 11. 15 lb.
12. 3 ft. 3 in. from the lighter particle.
13. 30 in. from the centre of the 5-lb. sphere. 14. 35 in.
15. $4\frac{4}{5}$ in. from the 6-oz. particle.
16. The line bisecting AB and CD at [illegible] its length from the middle point of AB.

EXAMPLES XXV. (PAGES 280-282.)

1. The C.G. of the bent wire divides the line joining the middle points of AB, BC in the ratio of 2 : 1.
2. 8 in. from A along the median through A.
3. Divides the median through the 2-lb. mass in the ratio of 7 : 5.
4. One-third of the weight of the plate.
5. Middle point of AG, where G is the C.G. of the triangle.
6. $8\frac{1}{4}$ in. from AB along the line bisecting AB and CD.
7. $13\frac{5}{7}$ in. from AB, $8\frac{4}{7}$ in. from BC. No.
8. $\frac{1}{6}$ OC along OC, where O is the centre of the hexagon.
9. $\frac{1}{7}$ distance from the centre to the fourth particle.
10. $\frac{15}{28}$ AD from AB along the line bisecting AB and CD.
11. $\frac{3}{8}$ AB from O along the line bisecting CD.
12. Middle point of GD, where D is the middle point of BC and G is the C.G. of the triangle.
13. $\frac{5}{6}$ AG from A along AG 14. The point D.
15. The point where the common axis crosses the junction of the two cylinders.
16. $\frac{1}{23}$ BD from O, the centre of the square, and lying in OD.
18. $14\frac{1}{21}$ in. from the end of the heaviest tube.
19. $\frac{1}{24}$ AB from O along the line through O, perpendicular to CD.
20. $\frac{1}{6}$ AB from O. 21. At [illegible] OC from O.
22. $AG = \frac{1}{13}(6 + 5\sqrt{3}) \times$ shorter side of rectangle.

23. 6·876 in. along the axis of the cylinder from the centre of the mouth.

24. 2$\frac{3}{5}$ ft.

25. 7$\frac{1}{3}$ in. from AB, 8$\frac{1}{3}$ in. from BC. 26. $\frac{1}{8}$ of the side of the square.

27. 3,000 ft.-lb. 28. 433,125,000 ft.-lb. 29. 792,000 ft.-lb.

30. $\frac{1}{6}r$ from centre of large circle, $\frac{5}{6}r$ from centre of smaller circle.

31. $\frac{4}{9}h$ from the diagonal (along the line bisecting the diagonal and the opposite side, the distance between which is h).

32. $\frac{3}{13}$ of the height of a triangle from the centre of the hexagon along the line bisecting the figure.

33. $\frac{45}{182}$ in. from the centre of the plate in the line joining it to the centre of the hole, and in the direction away from the centre of the hole.

34. $\frac{3}{4}$ distance from corner to C.G. of opposite face.

Examination Paper XIII. (Page 282, 283.)

1. See §§ 271, 272. 2. 100, 80, 60, and 60 lb. on A, B, C, D respectively.

3. See § 291. 4. 80 lb. 5. See §§ 293, 294. 6. 3$\frac{3}{4}$ in. from A.

7. See § 289. 8. See § 286. 9. 1$\frac{3}{5}$ in. from centre of larger circle.

10. 2 in. from the centre of the square in the line bisecting AB, CD.

Examples XXVI. (Pages 292-294.)

1. $\frac{4}{7}$ ft. towards the heavier wt. 2. 22$\frac{1}{2}$ lb. 3. 50 oz.

4. 2$\frac{6}{7}$ ft. from the 1-lb. wt.

5. The C.G. of the rod is vertically under the peg, and in the first case the rod is horizontal. In both cases, the strings are equally inclined to the vertical. Another position is with the rod vertical. 6. See § 298.

7. 12 lb.; the middle point is the C.G.

8. 2 oz.; 12 in. from the projecting end.

9. Each man will bear $\frac{1}{3}$. 10. The middle point.

11. 12 lb. 12. (i) See § 288. (ii) See § 297.

13. The position of equilibrium will not be affected. The opposite point is vertically below the point of suspension.

14. See § 302. With its square face on the plane, the block topples over when the height of the inclined plane is $\frac{3}{4}$ the base; with the 8-in. edge horizontal, when the inclination is 45°; and with the short edge horizontal and the 8-in. edge along the plane, when the height of the inclined plane is $\frac{5}{8}$ the base.

15. $\frac{1}{10}\sqrt{10}$ ft. 16. At the centre of the base of the hemisphere.
17. They will stand.
18. $\frac{8}{5}\sqrt{3}$ in. from BC, $2\sqrt{3}$ in. from CA, $\frac{12}{5}\sqrt{3}$ in. from AB.
19. $2\frac{38}{45}$ ft. from the 4-oz. particle.
20. 11 in. from CD, 15 in. from DA.
21. $\frac{11}{25}$AB from CD along the line bisecting AB, CD.
24. 40 lb., 32 lb., 24 lb.

Examples XXVII. (Pages 301-303.)

1. See § 315. 2. 15 lb. 3. 147 gm. 4. 23·04 oz.
5. 3*s*. $1\frac{1}{2}d$. 6. 14·4 lb. 7. 5 lb. 10 oz., 18 in., 20 in.
8. $\frac{38}{41}$ lb. 9. He loses $(a-b)^2W/ab$ lb. 10. See § 313.
11. No. 12. 5 : 5·099. 13. $34\frac{3}{8}$ in. from fulcrum.
14. 5 in., 13 in., 29 in., 61 in. 15. 16 lb., 4 lb.
16. $10\frac{3}{8}$ in. from the fulcrum.
17. 6 in., 4 in., and 3 in., from B, respectively.

Examination Paper XIV. (Page 303.)

1. See §§ 297, 295. 2. See § 301, Ex. 2. 3. 15°
4. See §§ 308, 319. 5. See § 312. 6. See § 314.
7. $34\frac{2}{15}$ oz. 8. See §§ 317, 318. 9. 18 lb., $12\sqrt{3}$ lb.
10. Twice the line joining A to the middle point of DE.